POLE-ZERO ANALYSIS OF DISTRIBUTED RADIO-FREQUENCY ACCELERATION SYSTEMS*

J. E. Katz and Q. A. Kerns

Lawrence Radiation Laboratory
University of California
Berkeley, California

Summary

It is shown that the frequency-dependent input impedance of a lossless distributed network may be represented by the frequencies of the input impedance poles and zeros. Then, given the network reactance at a single frequency other than that of a pole or zero, one may apply Foster's reactance theorem to determine the input impedance at any frequency.[1] Using the frequency-dependent impedance representation, one may calculate the response of the system to any periodic driving function.

General Considerations

In the design of radio-frequency acceleration systems, the problem of a power amplifier connected to a distant reactive load by means of a transmission line may occur. The response of such a system to a periodic current pulse that is short in duration with respect to an RF cycle is of interest to the designer.[2] For instance, a particular harmonic of the current waveform may excite an unwanted mode in the system being considered.[3]

It is necessary to know the frequency dependence of the input impedance of the two-terminal system in order to calculate the response.[1] For all but simple systems, the description of the impedance variation with frequency is usually a formidable task.

It is often relatively easier to specify or determine the frequencies at which the input impedance (for a lossless system) has an infinite value, called poles, and the frequencies at which the input impedance is zero, called zeros. This list of frequencies of the poles and zeros and the value of the input reactance at a given frequency completely specifies any two-terminal network composed of a finite number of reactive components, as shown by Foster.[1] Others have shown the equivalence between lumped constant circuits and transmission lines.[4, 5] These lumped-constant circuits are pure reactances for a lossless line.

The application of Foster's reactance theorem to a transmission-line system, which has an infinite set of poles and zeros, requires a consideration of the convergence of the infinite product formed. The convergence is proved in the attached appendix.

*Work done under auspices of the U. S. Atomic Energy Commission.

A Particular Case

To illustrate the analysis of a distributed system by the pole-zero method, we shall consider the case of a transmission line of uniform characteristic impedance Z_o, and length ℓ, which is shorted at the far end. The input impedance of this two-terminal system is obtained by considering the input-impedance poles and zeros. This result is compared with the input impedance (Z_{in}) derived from the transmission-line equations.

Applying the transmission-line equations, one obtains

$$Z_{in}(\omega) = jZ_o \tan \omega\ell/v \quad \text{[from transmission-line equation]} \qquad (1)$$

where v is the velocity of propagation of a fundamental mode (TEM) wave on the transmission line.

Input-impedance zeros (ζ) are obtained at $\omega = o$, and $\omega = n\pi v/\ell$, where $n = 1, 2, 3 \ldots$.

Input-impedance poles (ρ) are found at $\omega = \frac{\pi v}{\ell}(n - 1/2)$, where $n = 1, 2, 3 \ldots$.

Let $\zeta_n = n$, and $\rho_n = (n - 1/2)$.

The input impedance obtained by use of Foster's equation is

$$Z_{in}(\omega) = j\omega H \cdot \Pi_{n=1}^{\infty} \frac{[\omega^2 - (\zeta_n \pi v/\ell)^2]}{[\omega^2 - (\rho_n \pi v/\ell)^2]} \quad \text{[from Foster's equation]} \qquad (2)$$

where $\Pi_{n=1}^{\infty}$ indicates an infinite product with $n = 1, 2, 3 \ldots$. In order to show the equivalence between Foster's equation and the transmission-line equation, we let $s = \omega\ell/\pi v$; then

$$Z_{in}(s) = j\frac{\pi v s}{\ell} \cdot H' \cdot \Pi_{n=1}^{\infty} \frac{(s^2/\zeta_n^2 - 1)}{(s^2/\rho_n^2 - 1)}, \qquad (3)$$

where $H' = H \cdot \Pi_{n=1}^{\infty} (\zeta_n^2/\rho_n^2)$.

Now H may be evaluated by considering Z_{in} at a particular normalized frequency $s = s_o$, where $s_o \neq \zeta_n$ and $s_o \neq \rho_n$.

Solving Eq. (3) for H at $s = s_o$, we get

$$H = \frac{\ell Z_{in}(s_o)}{j\pi v s_o} \cdot \Pi_{n=1}^{\infty} \frac{\rho_n^2}{\zeta_n^2} \cdot \Pi_{n=1}^{\infty} \frac{(s_o^2/\rho_n^2 - 1)}{(s_o^2/\zeta_n^2 - 1)}. \qquad (4)$$

If we let $H' = \ell Z_o/v$, we obtain

$$Z_o = \frac{Z_{in}(s_o)}{js_o\pi} \cdot \Pi_{n=1}^{\infty} \frac{(s_o^2/\rho_n^2 - 1)}{(s_o^2/\zeta_n^2 - 1)} ; \qquad (5)$$

then from Fosters equation we see that

$$Z_{in}(s) = j\pi s Z_o \cdot \Pi_{n=1}^{\infty} \frac{(s^2/\zeta_n^2 - 1)}{(s^2/\rho_n^2 - 1)} \quad \text{[from Foster's equation]} . \qquad (6)$$

One may show that

$$\tan \pi s = \pi s \cdot \Pi_{n=1}^{\infty} \frac{(s^2/\zeta_n^2 - 1)}{(s^2/\rho_n^2 - 1)} = \tan \omega \ell / v \qquad (7)$$

by use of the Weierstrass factor theorem.[6]

Therefore, the above impedance expression is equivalent to that obtained from the transmission-line equation, and for the particular case shown

$Z_{in}(\omega)$ [from transmission-line equation] $\equiv Z_{in}(\omega)$ [from Foster's equation].

Conclusions

We have demonstrated how the continuous impedance expression of a lossless system may be represented by a set of discrete points. This representation for a lossless system is extremely useful, because most well-designed real systems have negligibly small losses. In practice, one would identify a finite number of impedance maxima and minima, and with the aid of a computer, generate the system voltage waveforms where desired in response to a given current input waveform.

Acknowledgments

The author's would like to express their appreciation to Dr. Loren P. Meissner of the Mathematics and Computing Section of this Laboratory; the appendix is taken from an unpublished memo contributed by him.

Appendix

When we attempt to extend Foster's formula[1]

$$Z_{in}(\omega) = j\omega H \cdot \Pi_{n=1}^{N} \frac{(\omega^2 - \zeta_n^2)}{(\omega^2 - \rho_n^2)} \qquad (A1)$$

(where ζ_n are the zeros of Z_{in} and ρ_n are the poles) to the infinite case ($N \to \infty$), we must study the convergence of the infinite product which, according to the factor theorem of Weierstrass, depends upon the convergence of the following infinite sum[6]:

$$\sum_{n=1}^{\infty} \left| \frac{\omega^2 - \zeta_n^2}{\omega^2 - \rho_n^2} - 1 \right| = \sum_{n=1}^{\infty} \left| \frac{\rho_n^2 - \zeta_n^2}{\omega^2 - \rho_n^2} \right| \qquad (A2)$$

We find that this product does not always converge. For example, if $\zeta_n = a \cdot n$ and $\rho_n = a \cdot (n - \frac{1}{2})$, we must consider

$$\sum_{n=1}^{\infty} \frac{(n^2 - n + \frac{1}{4} - n^2)a^2}{\omega^2 - (n^2 - n + \frac{1}{4})a^2} = \sum_{n=1}^{\infty} \frac{1 - \frac{1}{4}n}{(-\omega^2/a^2) + n - 1 + \frac{1}{4}n} , \qquad (A3)$$

where a is some arbitrary constant.

But, for any fixed ω (even if we exclude values of ω close to any of the poles), this series majorizes (i.e., termwise, it equals or exceeds in absolute value for sufficiently large n) the well-known divergent series

$$\sum_{n=1}^{\infty} \frac{k}{n} , \qquad (A4)$$

where $k = 1/2$, for example.

We now show that a revised form of Foster's formula can be extended to the infinite case under the following hypothesis, which includes many useful examples, including the one just considered. Let us write

$$Z_{in}(\omega) = j\omega H' \cdot \Pi_{n=1}^{N} \frac{(\omega^2/\zeta_n^2 - 1)}{(\omega^2/\rho_n^2 - 1)} , \qquad (A5)$$

where

$$H' = H \cdot \Pi_{n=1}^{N} \frac{\zeta_n^2}{\rho_n^2} . \qquad (A6)$$

Since H, and therefore H', is a constant (independent of ω), we see by considering a particular frequency $\omega = \omega_o$ that

$$H' = \frac{Z_{in}(\omega_o)}{j\omega_o} \cdot \Pi_{n=1}^{N} \frac{(\omega_o^2/\zeta_n^2 - 1)}{(\omega_o^2/\rho_n^2 - 1)} . \qquad (A7)$$

Now formulas (A5) and (A7) will provide a representation for $Z_{in}(\omega)$ in the infinite case ($N \to \infty$) also, provided that the infinite products are convergent. We must assume (in order for the factor theorem of Weierstrass to be valid) that neither the zeros (ζ_n) nor the poles (ρ_n) have a finite accumulation point. This means, in particular, that only a finite number of the poles and zeros have magnitude less than any given (large positive) R. We are interested in the uniform convergence of the formulas for all ω in some (finite) compact set Ω which excludes the poles: For instance, let Ω be a circle of finite radius centered at the origin, minus a small neighborhood of fixed size about each pole. As in Foster's paper, we assume that each pole and zero is of order one, and that symmetry allows us to combine the factors in pairs such as $\zeta_n = -\zeta_{-n}$ in order to form the connection between Eq. (A5) and the Weierstrass factor theorem.

The convergence of Eq. (A5) depends upon the convergence of the following infinite sum:

$$\sum_{n=1}^{\infty}\left|\frac{(\omega^2/\zeta_n^2 - 1)}{(\omega^2/\rho_n^2 - 1)} - 1\right| = \sum_{n=1}^{\infty}\left|\frac{\omega^2(\rho_n^2 - \zeta_n^2)}{\omega^2\zeta_n^2 - \zeta_n^2\rho_n^2}\right| . \quad (A8)$$

But in our compact set Ω, this sum Eq. (A8) is majorized uniformly (i.e., independent of ω, so long as $\omega \in \Omega$) by

$$\sum_{n=1}^{\infty} k \cdot \left|\frac{\rho_n^2 - \zeta_n^2}{\zeta_n^2\rho_n^2}\right| , \quad (A9)$$

where k is some number $k \geq 2 \cdot |\omega^2|_{max}$ on Ω. Let N be so large that all ρ_n for $n > N$ have a magnitude exceeding $2 \cdot |\omega|_{max}$ (which is possible since the poles have no finite accumulation point). Then $|\omega - \rho_n| > |\rho_n/2|$, so

$$|(\omega^2 - \rho_n^2)\zeta_n^2| = |\omega - \rho_n| \cdot |(\omega + \rho) \cdot \zeta_n^2| > |\rho_n/2| \cdot |\rho_n \cdot \zeta_n^2| = \frac{1}{2}\rho_n^2\zeta_n^2 , \quad (A10)$$

and

$$\left|\frac{\omega^2(\rho_n^2 - \zeta_n^2)}{(\omega^2 - \rho^2)\zeta_n^2}\right| \leq \frac{|\omega^2|_{max}|\rho_n^2 - \zeta_n^2|}{\frac{1}{2}|\rho_n^2\zeta_n^2|} \leq k\left|\frac{\rho_n^2 - \zeta_n^2}{\rho_n^2\zeta_n^2}\right| \quad (A11)$$

(as required) for $n > N$.

Hence, a sufficient condition for the validity of Eq. (A5) in the infinite case is the convergence of the sum

$$\sum_{n=1}^{\infty}\left|\frac{\rho_n^2 - \zeta_n^2}{\zeta_n^2\rho_n^2}\right| . \quad (A12)$$

This condition is readily verified in the example $\zeta_n = a \cdot n$, $\rho_n = a \cdot (n - \frac{1}{2})$: We have

$$\left|\frac{\rho_n^2 - \zeta_n^2}{\zeta_n^2\rho_n^2}\right| = \left|\frac{1 - \frac{1}{4}n}{n(n^2 - n + \frac{1}{4})}\right| , \quad (A13)$$

and the sum of (A12) is majorized by the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{(n-1)^3} .$$

References

1. Ronald A. Foster, A Reactance Theorem, Bell System Tech. J. 3, 259-267 (April 1924).
2. Q. A. Kerns and W. S. Flood, Stabilization of Accelerating Voltage Under High-Intensity Beam Loading, IEEE Trans. Nucl. Sci. NS-12, No. 3 (1965).
3. Q. A. Kerns, Cavity Resonator Systems for Supporting Nonsinusoidal Periodic Waveforms, UCRL-16242, June 30, 1965.
4. W. C. Johnson, Transmission Lines and Networks (McGraw-Hill Book Co., N. Y., 1950).
5. F. E. Terman, Electronics and Radio Engineering (McGraw-Hill Book Co., N. Y., 1932).
6. K. Knopp, Theory of Functions, Pt. II, Chap. 1 (Dover, N. Y., 1947).