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ON THE DESIGN OF AXIALLY SYMMETRIC ELECTRON GUNS*

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Introduction and Summary

In the design of high-perveance electron beam devices, it is necessary to evaluate the effects of the self-fields on the beam trajectories as well as the effects of any external electromagnetic fields that may be present. This paper describes a means whereby the self-consistent solution of Poisson's equation, the relativistic equations of electron motion, and the equation of charge continuity, all subject to specified boundary conditions, are determined by means of a digital computer program. In particular the method is applied to the case of an axially symmetric, solid-beam electron gun where the electrons are emitted with zero initial velocity from a space-charge-limited cathode.

In the first section of this paper, the equations of motion are developed using the relativistic Lagrangian and the well-known Euler-Lagrange equation. Cylindrical coordinates are employed, and the condition of axial symmetry is used to eliminate the azimuthal coordinate. The second section develops the equations describing the electromagnetic fields. The effect of space charge and the self-induced magnetic field due to the beam itself are included. The next section presents a brief description of the computer program and its requirements. The final section presents results of the calculation as applied to two electron-gun configurations. First, the calculation is applied to a 67.5° Pierce gun, and the results are compared with the theoretical results. Second, the calculation is applied to an electron-beam welding gun, and the results are compared with the experimental results. In general, the results show the manner in which an initially parallel beam subsequently diverges; they also show how an initially converging beam reduces to a minimum diameter and thereafter diverges.

1. Equations of Motion

The Lagrangian for an electron in an electromagnetic field is given[1] as

$$L = -m_0c^2\sqrt{1-\beta^2} + eV - e\vec{A}\cdot\vec{v} \qquad (1.1)$$

where

$$\beta^2 = v^2/c^2,\ \gamma = (1-\beta^2)^{-1/2},$$

and in cylindrical coordinates

$$\vec{v} = \dot{r}\vec{r} + r\dot{\theta}\vec{\theta} + \dot{z}\vec{z}\ .$$

In Eq. (1.1), m_0 is the rest mass of the electron with charge e, $\vec{v}$ is the velocity of the particle, c is the speed of light, V is the electric potential, and $\vec{A}$ is the magnetic vector potential. The "dots" denote d/dt. Using the Lagrangian above and specifying the magnetic and electric potentials, one can determine the equations of motion via the well-known Euler-Langrange equation.

For the case being considered here, an axially symmetric steady-state system, the electric and magnetic potential functions are independent of θ and t. However, the magnetic vector potential is allowed to have a θ-component. Moreover, θ is a cyclic coordinate and the canonically conjugate momentum is conserved; hence, a first integral of θ(t) can be written immediately as

$$m_0\gamma r^2\dot{\theta} - er\,A_\theta = \text{constant} \qquad (1.2)$$

where the constant depends upon the initial conditions. If at t = 0 the conditions are $r = r_0$, $\dot{\theta} = 0$, and $A_\theta = A_0$, then the result is

$$\dot{\theta} = \frac{\eta}{\gamma}\left(\frac{A_\theta}{r} - \frac{r_0}{r^2}A_0\right) \qquad (1.3)$$

where $\eta = e/m_0$. It should be noted that Eq. (1.3) is a form of the well-known Busch's theorem.[2] From Eq. (1.3) it is seen that a particle emitted in a magnetic field will obtain mechanical angular momentum when leaving the field. Similarly, a particle entering a magnetic field region will undergo a change in angular momentum; however, it will be restored to its initial value upon leaving the field.

The Euler-Lagrange relation yields the differential equation for the radial motion,

$$\frac{d}{dt}(\gamma\dot{r}) - \eta\frac{\partial V}{\partial r} - \eta\dot{z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) - \frac{\eta^2}{\gamma}\left(\frac{r_0}{r}A_0 - A_\theta\right)\left(\frac{r_0}{r^2}A_0 + \frac{\partial A_\theta}{\partial r}\right) = 0. \qquad (1.4)$$

Similarly, the Euler-Lagrange relation yields the differential equation for the axial motion,

$$\frac{d}{dt}(\gamma\dot{z}) - \eta\frac{\partial V}{\partial z} + \eta\dot{r}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) - \frac{\eta^2}{\gamma}\left(\frac{r_0}{r}A_0 - A_\theta\right)\frac{\partial A_\theta}{\partial z} = 0. \qquad (1.5)$$

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In addition

$$\frac{d}{dt}(\gamma\dot{r}) = \gamma\ddot{r} + \frac{K\dot{r}}{(1-\beta^2)^{3/2}} \tag{1.6}$$

and

$$\frac{d}{dt}(\gamma\dot{z}) = \gamma\ddot{z} + \frac{K\dot{z}}{(1-\beta^2)^{3/2}} \tag{1.7}$$

where

$$K = \frac{\dot{r}\,\ddot{r} + \dot{z}\,\ddot{z}}{c^2 + \eta^2\left(\frac{r_0}{r}A_0 - A_\theta\right)^2} - \eta^2\left(\frac{r_0}{r}A_0 - A_\theta\right) \times (c^2 - \dot{r}^2 - \dot{z}^2)\frac{\left[\dot{r}\left(\frac{r_0}{r^2}A_0 + \frac{\partial A_\theta}{\partial r}\right) + \dot{z}\,\frac{\partial A_\theta}{\partial z}\right]}{\left[c^2 + \eta^2\left(\frac{r_0}{r}A_0 - A_\theta\right)^2\right]^2} . \tag{1.8}$$

The velocity may be determined by

$$v^2 = \frac{\dot{r}^2 + \eta^2\left(\frac{r_0}{r}A_0 - A_\theta\right)^2 + \dot{z}^2}{1 + \frac{\eta^2}{c^2}\left(\frac{r_0}{r}A_0 - A_\theta\right)^2} . \tag{1.9}$$

The above equations describe the general, relativistic trajectories for an electron in a cylindrically symmetric electromagnetic field. It should be pointed out that the equations are highly nonlinear, and analytic solutions to interesting problems would for all practical purposes be impossible.

2. Electromagnetic Fields

The magnetic field, $\vec{B}$, may be determined from the magnetic vector potential, $\vec{A}$, by

$$\vec{B} = \nabla \times \vec{A} . \tag{2.1}$$

For the case of cylindrical symmetry, the components of $\vec{B}$ can be written from Eq. (2.1) as

$$B_r = -\frac{\partial A_\theta}{\partial z} , \tag{2.2a}$$

$$B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} , \tag{2.2b}$$

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta) . \tag{2.2c}$$

Identify B_θ as the self-induced, θ-directed magnetic field due to the beam itself. For a long tube of current, B_θ may be determined from

$$\int B_\theta\, d\ell = \mu_0 \int j_z(r) r\, dr\, d\theta$$

where j_z is the z-component of the current density. Since $j_z = j(\dot{z}/v)$, then the result for B_θ may be written

$$B_\theta = \frac{\mu_0}{r}\int_0^r j\,\frac{\dot{z}}{v}\,r\,dr \tag{2.3}$$

where j is the total current density, $\dot{z}$ is the z-component of the particle velocity, and v is total particle velocity as given by Eq. (1.9). Equation (2.3) may be integrated in order to determine the self-induced magnetic field at a point r within the beam.

The θ-component of the magnetic vector potential is defined via Eq. (2.2c) as

$$A_\theta = \frac{1}{r}\int_0^r rB_z(r)\,dr.$$

If B_z consists of a dc external applied field plus a self-induced axial field component, A_θ can then be written

$$A_\theta = r\,\frac{B_z(0)}{2} + \frac{1}{r}\int_0^r rb_z(r)\,dr. \tag{2.4}$$

In Eq. (2.4) B_z is the applied dc field and is assumed to be constant over the radius of the beam; $b_z(r)$ is the self-induced axial magnetic field.

This self-induced axial magnetic field may be determined by

$$b_z(r) = \mu_0\int_r^R j_\theta\,dr \tag{2.5}$$

where R is the outer radius of the beam and j_θ is the θ-component of the current density. The relation (2.5) holds in the approximation of no radial motion and no variation of j_θ with z. Since the numerical procedure will be dealing with calculations at discrete points, the approximation will be valid if the distance between successive points is sufficiently small. The other self-induced components of the magnetic field are at present ignored in the calculation.

The electric field is determined via the solution to Poisson's equation which may be written for a cylindrically symmetric system as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial z^2} = -\rho/\epsilon_0 \tag{2.6}$$

where ρ is the space charge density and ϵ_0 is the permittivity of free space; ρ may be determined in terms of the convection current density and velocity at any point by

$$\rho = j/v. \tag{2.7}$$

The space charge density may be calculated from Eq. (2.7), and this result can then be substituted in Eq. (2.6) to calculate the required electric fields.

Space-charge neutralization effects may be studied by multiplying the space charge term on the right of Eq. (2.6) by some factor, $f\ (0 \leq f \leq 1)$. In this manner the electric fields may be determined for neutralized beams ($f = 0$), for partially neutralized beams ($0 < f < 1$), and for completely unneutralized beams ($f = 1$).

Since the problem to be considered is the steady-state flow problem, the total beam current must remain constant. That is, the continuity equation holds,

$$\cdot\ j = 0. \tag{2.8}$$

However, the current density, j, is a function of radius. If the beam is made up of a number of discrete rays each of radial width Δr, then the conservation of current requirement leads to

$$j = \frac{r_0}{r} j_0 \tag{2.9}$$

where j is the current density at a radius r and j_0 is the initial current density at r_0.

It will be assumed that the electrons are emitted with zero initial velocity from a space-charge-limited cathode. Therefore, the emission current density is determined using Child's law,

$$j = \frac{4}{9} \epsilon_0 \sqrt{2\eta}\ \frac{V^{3/2}}{d^2}\ , \tag{2.10}$$

where d is the perpendicular distance from the cathode to the field point with voltage V.

3. Computer Program

The numerical procedure is an extension of the one given by Boers.[3] The differential equations are reduced to a set of finite difference equations. The electrode boundaries are laid out on a matrix of points covering the region of interest. Starting with an assumed space-charge density distribution $\rho(r,z)$ (an initial guess), Poisson's equation is solved via a relaxation technique subject to the specified boundary conditions. The magnetic field quantities are determined by the appropriate calculation as given in the previous section. The initial conditions are determined for the case of an emitting surface. Knowing the electromagnetic field quantities and the entrance conditions, one can solve the equations of motion for the trajectories. New values of space charge density are then calculated from the resultant trajectories. The entire process is then repeated until a convergence criterion is satisfied. Thermal effects and direct particle-particle collisions are not considered; however, crossing of trajectories within the beam is permitted.

The program is written in FORTRAN IV for the CDC 6600 and requires approximately 45,000 memory locations for execution. A typical problem requires an execution time of approximately 10 minutes. Obviously, the execution time depends upon the care used in setting up the problem, the convergence criterion, the complexity of the electrode boundaries, the size of the matrix, and the like. When the convergence criterion is achieved, the results are printed and, if desired, plotted directly.

4. Results

The calculation described above has been applied to a 67.5° Pierce gun as shown schematically in Fig. 1. In addition several representative trajectories and equipotentials are shown in the figure. The current density determined by the calculation deviates from the theoretical, uniform value predicted by Child's law by approximately 13% at the edge of the cathode and by less than 0.7% in the central region of the cathode. The trajectories are slightly curved in the accelerating region, a result due partly to the hole in the anode (that is, the anode is not a solid collector plate) and partly to the self-induced magnetic field that is present.

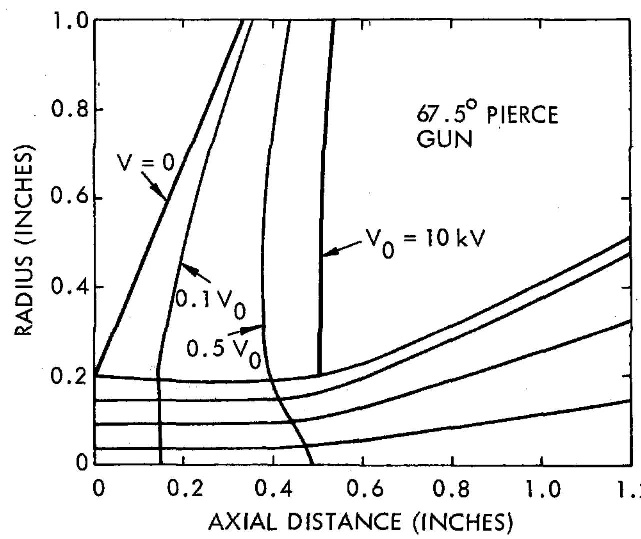

Fig. 1. Trajectories and equipotentials for a 67.5° Pierce gun.

A small electron-beam welder[4] has been experimentally developed at LRL. Figures 2 and 3 show a schematic representation of the electron gun for two cases. In Fig. 2 is shown the case for a space-charge-limited cathode with no space charge neutralization present. Several representative trajectories and equipotential contours are shown in the figure. In this case the beam current is computed to be 0.28 A at an anode voltage of 10 kV for a perveance of 2.8×10^{-7}.

Figure 3 presents results for the same case as in Fig. 2 but with a fully neutralized beam space-charge. In this configuration the beam

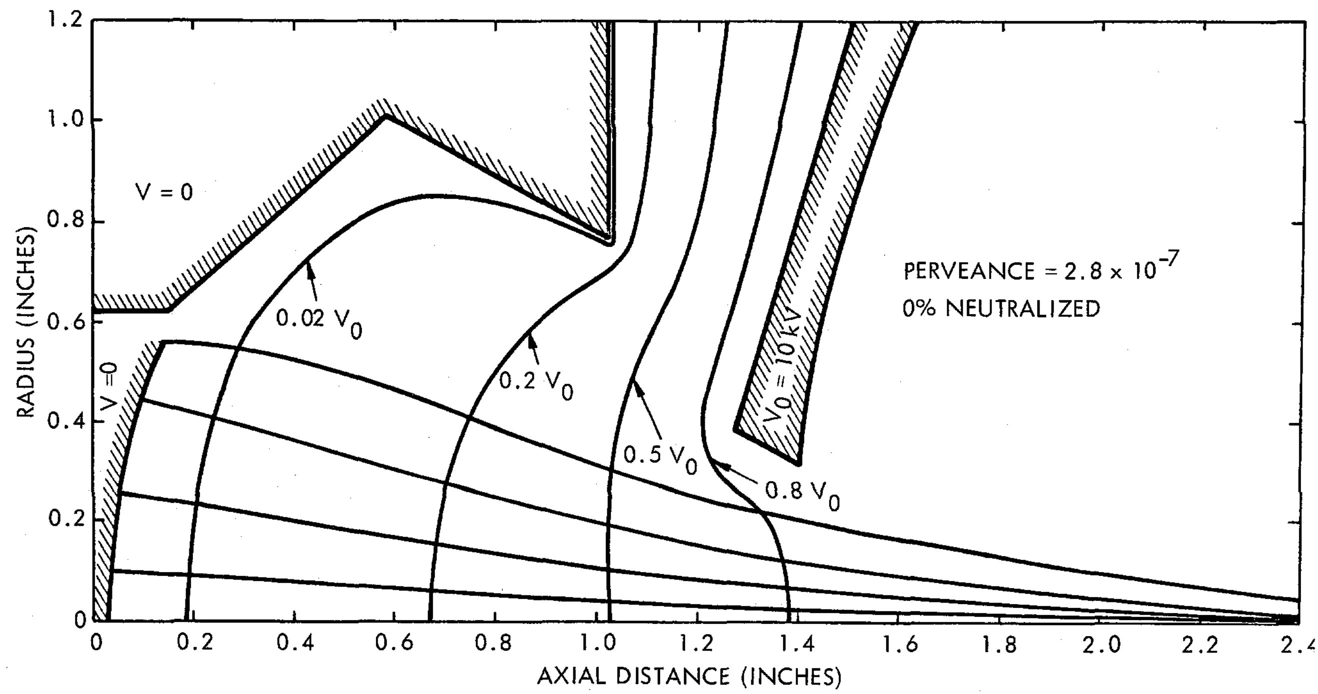

Fig. 2. Trajectories and equipotentials for an electron-beam welding gun. No space-charge neutralization is included.

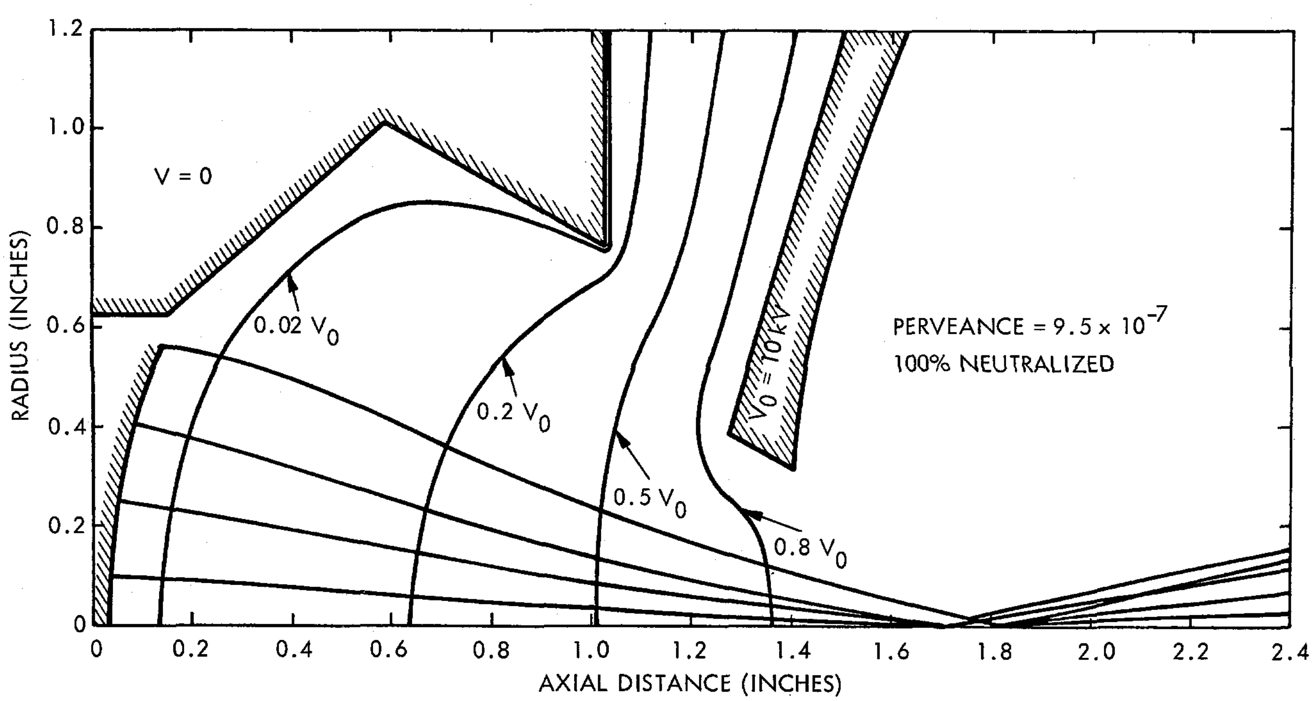

Fig. 3. Trajectories and equipotentials for an electron-beam welding gun. Full space-charge neutralization is included.

current is computed to be 0.95 A at 10 kV anode voltage for a perveance of 9.5×10^{-7}. Several representative trajectories as well as the beam edge trajectory are shown in the figure.

Comparing Figs. 2 and 3, one can see that the space charge forces tend to prevent excessive trajectory crossing. In addition the beam waist (minimum beam diameter) is moved farther away from the cathode for the case of no neutralization. Comparing the positions of the equipotentials for the two cases, one can see that the position of a given equipotential is farther away from the cathode for the unneutralized case compared to the neutralized case. This shift in the position of the equipotential is due to the negative space-charge potential generated by an unneutralized beam.

These computed results agree reasonably well with the actual electron gun data. Since the electron beam welder operates dc in a vacuum of the order of 10^{-5}-10^{-4} mm Hg and is bombarding a target, it is reasonable to expect that some degree of neutralization is present. In operation the welder has a cathode current of approximately 0.7 A at an anode voltage of 10 kV for a perveance of 0.7×10^{-7}. This would indicate that some degree of neutralization is present. The beam properties such as the position of the beam waist also tend to verify this conclusion and in fact suggest that the beam is made up of an unneutralized core surrounded by a somewhat neutralized cloud.

References

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[2] J. R. Pierce, Theory and Design of Electron Beams (D. Van Nostrand Co., New York, 1954), second edition.

[3] J. E. Boers, IEEE Trans. Elect. Dev. ED-12, pp. 425-435, July 1965.

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