# CALCULATION OF SPACE CHARGE EFFECTS IN BEAM TRANSPORT

R. C. Catura
Princeton-Pennsylvania Accelerator
Princeton, New Jersey

## Summary

Methods of calculating space charge effects which are presently available have as their basis the assumption of uniform charge density over the beam cross section. In design of a high intensity beam transport system, it is important to know how sensitive parameters of the design are to this assumption. This question has been investigated by propagating phase plots along the beam axis for space charge fields from both a uniform charge density and a charge density obeying a normal distribution. Normalization was such that the field from the uniform density was a least-squares fit to the field from the charge density of normal distribution. Results of computations for 20 mA of 3.0 MeV protons indicate that while the phase plot showed considerable distortion from effects of the normal charge density, parameters of the plot which effect design of a transport system were not changed appreciably. From these calculations it seems that unless close matching of an acceptance phase plot is imperative, design of a transport system is relatively insensitive to charge distribution in the beam.

## Introduction

In the transport of beam between pre-accelerator and accelerator space charge begins to play an appreciable role in particle dynamics for currents in the milliampere range. Calculation of this effect is complicated by the coupling which exists between the beam phase plot and its equations of transformation. This occurs because forces introduced by space charge depend on the density distribution in phase space. Because of difficulty in determining the electric field in an arbitrary charge distribution calculations can be readily made only for certain phase densities. One requires the phase density and the electric field from space charge to be such as to always produce a linear transformation of the phase plot as the beam propagates through the transport system. This assures, for example, that an elliptical phase plot maps into another ellipse but restricts the electric field to being linear with displacement from the beam axis.

Differential equations describing dynamics of a beam envelope which include the effects of space charge and emittance have been derived by Kapchinskij and Vladimirskij[1] and also Walsh[2]. In both derivations, the charge density is assumed to be uniform over the beam cross section and the phase distribution such as to maintain this uniformity as the beam propagates. Kapchinskij and Vladimirskij assume a transverse phase distribution in which all points lie on the surface of a four dimensional ellipsoid with uniform density. As Walsh points out, this assumption means that at a point where the beam passes through a minimum in radius the velocity vectors of all particles at a given point lie on the surface of a cone. The opening angle of the cone is zero at the beam surface and reaches a maximum at the center. Their assumption also leads to a uniform phase density in the transverse two dimensional phase plots. Walsh chooses a more general phase distribution, obtains an expression for its Fourier transform, but is unable to invert it.

It is apparent that a rather special phase distribution is necessary to make differential equations describing beam propagation self-consistent. Since the exact phase distribution necessary to maintain self-consistency may not occur physically, it is important to inquire the effect a non-uniform charge density has on beam dynamics. This is of particular interest if the assumptions described above have been used in calculations for design of a beam transport system. The question then posed is whether the effect of a non-uniform charge density, which may occur in the physical beam, acts as a perturbation, or if it will be large enough to change design parameters. This question was studied by observing differences in the phase plot as the beam propagates for electric fields resulting from a charge density which is uniform and one which follows a normal distribution.

## Method of Calculation

A Cartesian coordinate system is used for calculation where z is along the beam axis with x and y being transverse coordinates. The beam is assumed to have circular symmetry in the x, y plane so the electric field from space charge is in a radial direction. Only the portion of the y, $\dot{y}$ (derivative of y with respect to z) phase plot is considered for which x and $\dot{x}$ equal zero. Observe that this limited phase plot has the same boundary as the entire plot for y, $\dot{y}$, since at a given y, the value of $\dot{y}$ attains its maximum if $\dot{x} = 0$. For the phase distribution assumed in reference (1), the limited phase plot is exactly the boundary (and no more) of the entire y, $\dot{y}$ phase plot. Using the limited phase plot imposes no restrictions for the present application since only a comparison of phase boundaries is of interest. A limited phase plot is chosen because if x equals zero, the radial coordinate r and y are identical; and, if $\dot{x}$ equals zero, they remain identical as the beam propagates. The radial electric field then lies along the y direction and the differential equation for y motion is simplified. In addition, measurements of y, $\dot{y}$ phase boundaries with x equals zero have been made by van Steenbergen[3] and may be qualitatively compared with results of the present calculations.

Employing assumptions made above the differential equation describing y-motion of a particle with mass, m, and charge, e, under action of a radial electric field, $E(r,r_o)$, due to space charge is:

$$\frac{d^2y}{dz^2} = \frac{eE(r,r_o)}{mv^2} \qquad (1)$$

Where $r_o$ is the beam radius, and v is the particle's velocity. For a uniform charge distribution with density s per unit length along the beam axis:

$$E(r,r_o) = \frac{2sr}{r_o^2} \qquad (2)$$

If the normal charge distribution:

$$\rho(r) = \frac{s}{\pi\sigma^2} e^{-r^2/\sigma^2}$$

is considered, the resulting electric field is given by:

$$E(r,\sigma) = \frac{2s}{r} (1-e^{-r^2/\sigma^2}). \qquad (3)$$

In order to compare solutions of the differential equation for the electric fields given by equations (2) and (3), it is necessary to relate the parameters σ and $r_o$.

The normalized function:

$$W = \frac{\sigma E(r,\sigma)}{2s}$$

is fitted to a straight line passing through the origin by a method of least-squares weighted by the charge density. The range of values for r/σ considered in the fit lies between 0 and 1.6 and includes 92% of the charge within it. The function W and the least-squares straight line are shown in Figure 1. As a definition of σ, the linear function for E(r,σ) resulting from the least-squares fit is required to be equal to the electric field for a uniform charge density of radius $r_o$. The value of σ obtained from this equality is .79 $r_o$. With this normalization the field for the uniform charge density is a least-squares fit to the electric field of the normal distribution.

Solutions of the differential equation can now be investigated for electric fields defined by equations (2) and (3). Equation (1) describes motion of a single particle while the value of $r_o$, which is also a function of z, depends on motion of the entire beam. Numerical solution of the differential equation is accomplished in the following manner: A computer program called "BEAM", described in paper II-16 of these proceedings, was used to calculate the beam radius $r_o$ at regular intervals, Δz, along the z axis. This program solves the differential equation governing the beam envelope and employs the assumptions of Kapchinskij and Vladimirskij described earlier. Only the situation where r and y are identical will be considered henceforth and the two will be used interchangably. Using the value of $r_o$ obtained from the computer program at a particular z one may evaluate $E(y,r_o)$ in equations (2) and (3) for any given y. This value of $r_o$ is correct for the uniform charge density and it will be assumed that the radius of the normal distribution does not differ appreciably. It will turn out that this is a good assumption. It will also be assumed that distortions of the phase plot do not effect the normal charge distribution. The size of Δz is chosen small enough so that variation of $E(y,r_o)$ over this interval may be neglected. The differential equation (1) becomes that for a uniformly accelerated particle over the increment Δz and may be readily solved for y and $\dot{y}$. The process is repeated for the new value of y and the corresponding value of $r_o$ at z + Δz. In this way values of y and $\dot{y}$ for any point on the phase plot may be propagated along the z axis. A consequence of Liouville's theorem is that points on the boundary of a phase plot must remain on the boundary. The boundary of a phase plot may then be mapped along the z axis point by point using the methods described above. A computer program was written to perform these operations for electric fields from both the uniform and normal charge densities. Values of $r_o$ as calculated by "BEAM" were supplied the program as input data.

Calculations were made for 3 phase plots with emittances of 0.8, 0.4 and 0.2 mrad-cm. The boundaries of the phase plots at z = 0 were taken as ellipses oriented with respect to the y, $\dot{y}$ axes in such a way that the beam was focused at z = 108" if space charge effects were neglected. These plots differed only in that the semi-axes were reduced by $\sqrt{2}$ for each decrease in emittance. Using a value of Δz = .5", the computer program was used to propagate these phase plots point by point a distance of 300". In each case the beam was taken to be 20 mA of 3.0 MeV protons. Results of these computations are shown in Figures 2 to 7. In each figure the solid line denotes the phase plot for a uniform charge density, the broken line that for a normal distribution and the solid circles show the phase plot in the absenceof space charge effects. Validity of this method for propagating the phase plots may be checked by comparing the plots for a uniform charge distribution at 300" with those which may be obtained from "BEAM". The program "BEAM" uses a much smaller mesh, Δz, than .5" and propagates the phase boundary by a linear transformation. This comparison shows that phase boundaries obtained by the two methods differ by less than 2% at any point on the ellipses for the case with an emittance of 0.2 mrad-cm., in which space charge effects are largest.

## Discussion

Figures 2 to 4 show the phase plots near (within the mesh size of 0.5") the point where the beam with uniform charge density undergoes a minimum in radius. Since the same beam current of 20 mA is assumed in each case space charge should have least effect for large emittance because the beam is larger in radius. An increase in distortion of the phase plot for the normal distribution is apparent as the emittance decreases, but the beam radius is not effected greatly. The general effect of the distortion seems to be a shift of phase area to larger radii. One can also observe from these graphs a shift in the point of minimum radius whose sign depends on the magnitude of the space charge effect. The beam without space charge focuses at z = 108". For the largest emittance, space charge is essentially a perturbation and the focal point is shifted 4" to a larger value of z as indicated in the caption. As space charge becomes more pronounced the shift changes sign and the point of minimum radius occurs earlier than z = 108". This may be understood by observing that in a converging beam, space charge, acting as a perturbation, tends to reduce the velocity of convergence so the beam takes longer to focus. If the space charge effect becomes very large the radial velocity of the beam envelope will change sign (point of minimum radius) before the focal point without space charge is reached.

In Figures 5 to 7 the phase plots are shown at z = 300". Since the graphs are symmetrical only positive values of $\dot{y}$ are shown to increase details of the phase plots. Again, the displacement of phase area to larger radii for the normal charge distribution is apparent, especially for the case with an emittance of 0.2 mrad-cm. This is a very undesirable effect if it is necessary to closely match the acceptance phase plot of an accelerator or if an aperture is present in the system. As before, the radius of the beam is not appreciably changed by the non-uniform charge density. One may qualitatively understand these effects by reference to Figure 1. At small radii the electric field is larger for the normal distribution than for the uniform density of least-squares fit. This produces a relatively larger radial acceleration in the case of the normal charge distribution and tends to populate phase space at larger radii. At larger radii the opposite effect is true and the least-squares fit overestimates the electric field for the normal distribution. One would then expect phase space to become more heavily populated at intermediate radii as tends to be the case in Figure 5. As the space charge effect increases with decreasing emittance the relatively larger electric field for the normal distribution at small radii seems to dominate as shown in Figures 6 and 7. The radius of the normal distribution in these figures tends to be slightly larger than that for the uniform distribution.

Emittance phase plots for a pre-accelerator measured by van Steenbergen[3] show a tendency for phase area to be displaced to large radii. He considers distortions in the phase plots to be from aberrations in an electrostatic lens system but also suggests space charge as a contributing factor. Present calculations show that space charge effects of a non-uniform charge density can cause this general type of distortion.

A point of further interest is the trajectory of individual particles within the beam. The point on the phase boundary marked with an X in Figures 4 and 7 was at a radius of approximately $0.1r_0$ at z = 0. In Figure 7 it has moved to approximately $0.8r_0$. If the beam is confined by quadrupole focusing over a long transport system individual particles may shift their relative radial position sufficiently that effects of a non uniform electric field tend to average out.

Although phase plots at z = 300" for uniform and non-uniform charge densities differ in shape, their radii and the angle their axes make with the y axis are nearly the same. This is an encouraging result from the standpoint of designing a beam transport system. If such a system were designed assuming a uniform charge density one could also expect to accommodate a beam with normally distributed charge. From these calculations it seems that design of a transport system is relatively insensitive to charge distribution in the beam unless close matching of an acceptance phase plot is imperative.

The author wishes to acknowledge Mr. A. Passner for his aid in programming and in carrying out computations involved in this work.

## References

1. I. M. Kapchinskij and V. V. Vladimirskij, Conference on High-Energy Accelerators and Instrumentation, Session 3, 274, C.E.R.N. Geneva (1959).

2. T. R. Walsh, Journal of Nuclear Energy, Part C, 5, 17, (1963).

3. A. van Steenbergen, Brookhaven National Laboratory, AGS Internal Report AvS-2 (August 1962).

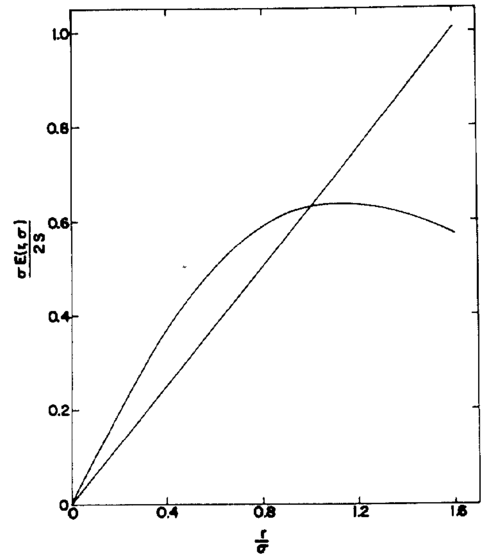

Fig. 1. Electric field vs radius. The normal charge distribution is given by curved line while the least-squares fit is shown by a straight line. Units are dimensionless.

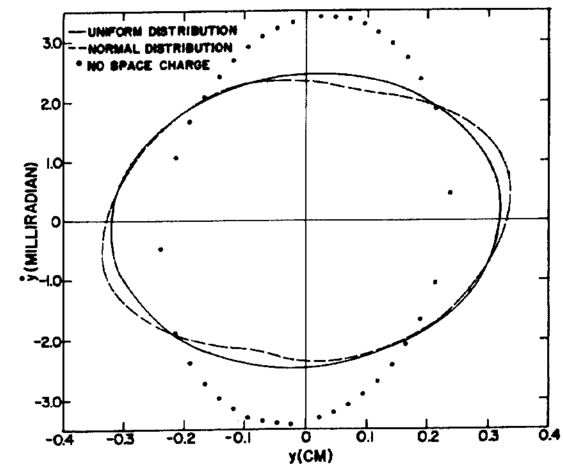


Fig. 2. Phase plots for emittance of 0.8 mrad-cm at z = 112" (minimum radius).

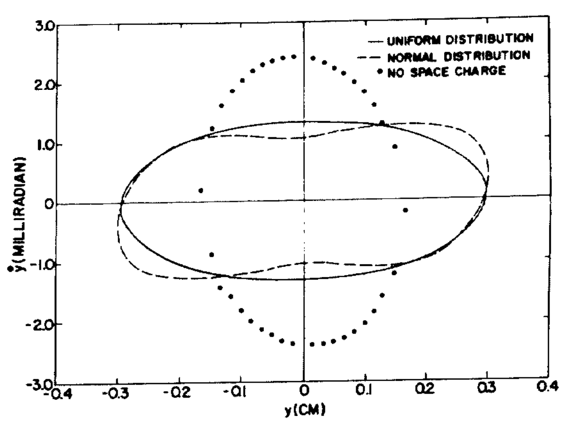


Fig. 3. Phase plots for emittance of 0.8 mrad-cm at z = 106" (minimum radius).

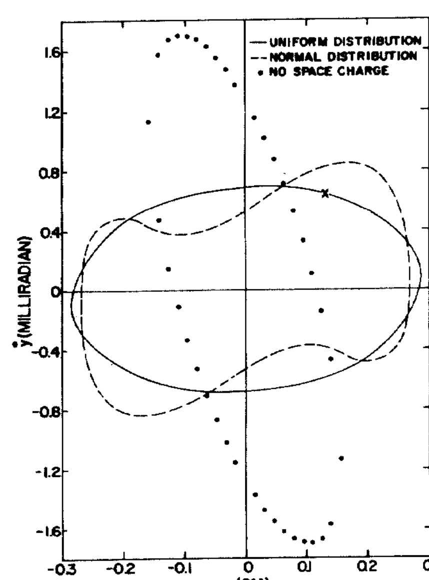


Fig. 4. Phase plots for emittance of 0.2 mrad-cm at z = 84" (minimum radius).

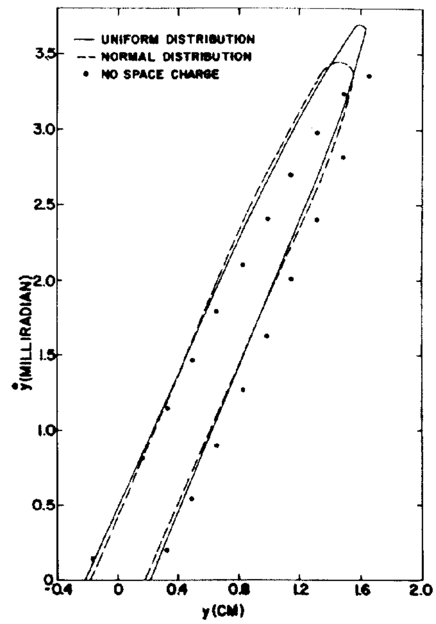


Fig. 5. Semi-phase plots for emittance of 0.8 mrad-cm at z = 300".

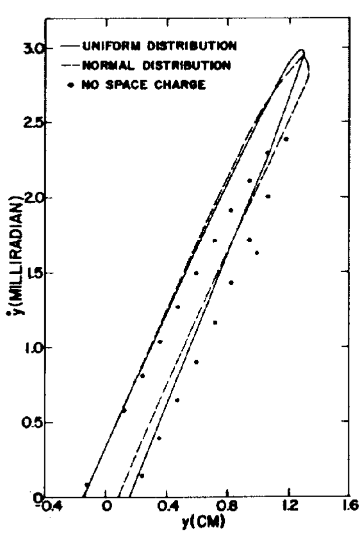


Fig. 6. Semi-phase plots for emittance of 0.4 mrad-cm at z = 300".

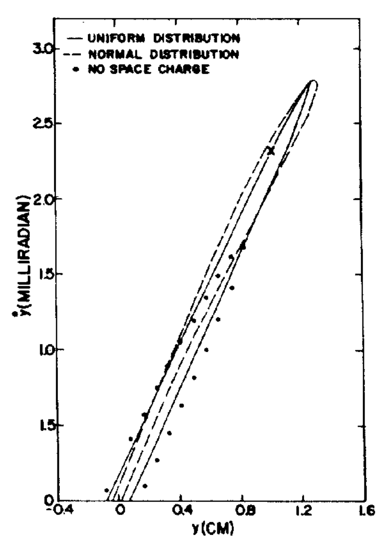


Fig. 7. Semi-phase plots for emittance of 0.2 mrad-cm at z = 300".