BEAM INSTABILITIES IN CIRCULAR ACCELERATORS*

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I. Introduction

The particles in circular accelerators must make between hundreds and millions of revolutions in the acceleration process. The accelerator must therefore be designed so that the orbits are stable, both in transverse and longitudinal phase space. Traditionally fields in accelerators have been designed so that the motions of a single particle in the designed electromagnetic field would be stable and lead to the desired energy. It was soon appreciated, however, that this is not sufficient: actual hardware never conforms exactly to an ideal design, and one has to worry about the effects of deviations of the field from the ideal — whether caused by magnet misalignments, constructional tolerances, or radiofrequency noise — on the stability of the particle motions. It was found that the effects of such errors could be minimized, but not eliminated, by designing the orbit parameters so that the frequencies of betatron oscillations were as far as possible from the integral and half-integral resonance values

$$\nu = k$$
$$\nu = k + \tfrac{1}{2}$$

where ν is the ratio of betatron oscillation frequency to revolution frequency. In the vicinity of each resonance there is a "stop band", i.e. a range of parameters for which the motion is unstable; the width of the stop band depends on the magnitude of the field gradient errors. By avoiding the resonances and building and erecting the machine with reasonably close tolerances, one can, and does, ensure that the particles remain in stable orbits close to the ideal ones. The resonances are avoided by designing the magnetic field so that the operating point lies close to $\nu = k \pm \tfrac{1}{4}$. Stability should then be maintained as long as extraneous fields, caused by sources not incorporated in the design, change ν by less than $\tfrac{1}{4}$.

Such extraneous fields may be caused by variations in magnet properties (especially at the low fields of injection and at saturation). Such errors can be compensated by poleface windings or multipole lenses. But there is another "extraneous" field, namely the field generated by the beam itself. In recent years the beam intensity of large proton accelerators has been brought up to the order of 100 milliamperes of circulating beam, and this can seriously alter the forces felt by the particle. Not only can these space-charge forces alter the value of ν until it is shifted to a resonance, but several modes of plasma oscillations can occur which may have exponentially growing amplitudes.

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II. Space-Charge Detuning

The space-charge detuning is easily calculated for a beam of uniform density having an elliptical cross section. The effect is

$$\Delta\nu = -\frac{N}{\pi\nu B}\,\frac{r_o R}{b(a+b)}\,\frac{1}{\beta^2\gamma}\,(1-\beta^2) \qquad (1)$$

where N is the number of particles in the beam, R is the orbit radius, $r_o = e^2/Mc^2$ the classical electrostatic particle radius, a and b the major and minor semi-axes of the elliptical cross section, and B is the bunching factor, i.e. the fraction of the circumference occupied by the beam. The factor $1-\beta^2$ arises from the combination of electric repulsion and magnetic attraction between the particles: the strength of the latter is β^2 times the former.

Laslett[1] has worked out modifications to this formula arising from the image fields generated in the walls of the vacuum chamber and the poles of the magnet. The main effect of these modifications is to add terms which depend on the chamber dimensions rather than the beam dimensions, and which decrease with increasing energy only as $1/\beta^2\gamma$ rather than $1/\beta^2\gamma^3$. Therefore the Laslett corrections are mainly of importance when the particle energy is relativistic.

The formula (1) predicts the shift in frequency of <u>incoherent</u> oscillations, i.e. of oscillations of individual particles about the center of the beam. Instability may be expected to arise when the frequency is shifted to the nearest "stop band", i.e. to the nearest half-integral value. In addition, there is a shift in the frequency of <u>coherent</u> oscillations, i.e. oscillations of the center of the beam. These will cause beam blowup only if the shifted frequency is integral. Laslett[1] has derived expressions for the coherent frequency shift; these depend only on the chamber and gap dimensions and not on the beam dimension; the shift goes with energy about as $1/\beta^2\gamma$. Thus, coherent frequency shifts are small compared to incoherent ones at low energies, but become comparable at relativistic energies.

Using these equations we can estimate "space-charge limits" for existing and projected accelerators. These will, of course, depend on the assumed beam dimensions and bunching factors. For the Brookhaven AGS we know that $a \approx 3.0$ cm, $b \approx 1.5$ cm, bunching factor $\approx 1/5$ (shortly after injection); we have accelerated as many as 1.4×10^{12} particles. For this beam we compute

(with $R = 128$ m, $\nu = 8.75$, $\beta^2\gamma = 0.1$)

$$\Delta\nu = 0.75 ,$$

three times the supposed limit!

The explanation for this discrepancy may be:

(a) The bunching factor equals 1/5 only during a few revolutions; the average value around injection time is more nearly twice that.

(b) A calculation by L. Smith[2] indicates that the actual frequency shift of the particles in a beam is effectively only half that given by (1), largely because the restoring force constant is modulated by the oscillations in the beam cross section. While his calculation is not rigorous, the experimental result may be taken to support it.

III. Plasma Effects

The first plasma instability to be observed in particle accelerators was the "negative-mass" longitudinal instability.[3-5] Here a beam of particles circulates in an accelerator magnet at an energy above the transition energy, so that the revolution frequency in the given field is a decreasing function of energy. This feature is called "negative mass" because it means that if a particle receives a forward force its angular acceleration is backward. Under these conditions, the space-charge repulsion between particles can cause exponentially growing fluctuations in the beam density. To see this qualitatively, consider just two particles. Their repulsion causes the forward one to receive a forward force, and therefore to slow down azimuthally; the rear particle thus comes closer to it, and the forces become stronger.

Analysis[3-5] shows that this unstable growth of fluctuations is inhibited by Landau damping arising from an energy spread in the beam. The criterion for stability turns out to be

$$\Delta E > 2f \left(- \frac{N e^2}{2\pi R f \frac{\partial f}{\partial E} \gamma^2} \left(1 + 2 \ln \frac{b}{a}\right)\right)^{\frac{1}{2}} \tag{2}$$

for a beam of radius a in a circular vacuum chamber of radius b. The negative sign signifies that $\partial f/\partial E$ must be negative for this instability to occur.

This instability has been observed experimentally in numerous accelerators, including the Cosmotron,[6] the MURA 50-MeV accelerator,[7] and the "Saturne" synchrotron at Saclay.[8] The agreement with theory is reasonably good in most cases. However, at MURA[7] it was found that a similar instability can arise at injection, which is well below transition energy, i.e. with $\partial f/\partial E$ positive, where the negative-mass theory predicts stability. Furthermore, coherent transverse oscillations were observed to occur with increasing amplitude in the MURA accelerator,[7] in the Cosmotron,[9] and in the Stanford electron storage rings.

Theories of these effects have been formulated by Neil and Sessler[10] for the longitudinal instability and by Laslett, Neil and Sessler[11] for the transverse instability. In both cases, a coherent disturbance in the beam induces image charges and currents in the wall of the vacuum chamber, which in turn generate fields that exert forces on the beam. Because of the finite resistivity of the walls, these forces are shifted in phase with respect to the disturbance in the beam. For those Fourier components which correspond to waves of field traveling in the direction of the particles but slower than the particles, the phase difference is of such a sign as to enhance the disturbance of the beam and therefore lead to instability.

These instabilities are again suppressed by Landau damping caused by spreads in the energy or the intrinsic oscillation frequency of the particles in the beam. In the case of the longitudinal resistive instability, the criterion for stability turns out to be very nearly the same as (2) (with the sign of $\partial f/\partial E$ reversed), except for a factor that depends on the form of the distribution in energy and, very weakly, on the resistivity of the wall material. This case will be discussed in more detail by R.L. Pease at this conference.[12]

A simplified analysis of the transverse resistive instability may be given as follows: Suppose we have an ensemble of N particles, the k^{th} one of which has an intrinsic oscillation frequency $\nu_k\Omega$ (Ω = frequency of revolution, assumed the same for all particles). Because of the interaction between the particles and the walls, we look for a normal mode in which all particles oscillate with the same frequency $\nu_o\Omega$, and attempt to find the conditions for which ν_o is real; these correspond to the threshold for instability. Let the transverse motion of the k^{th} particle be

$$z_k = \zeta_k \, e^{i(\nu_o\Omega t + \varphi_k)} \tag{3}$$

and consider a mode such that the phases φ_k are correlated so that the overall dipole moment is of the form

$$e\Sigma z_k = e N \xi \, e^{i(n\theta - \omega t)} \tag{4}$$

$$\omega = (n - \nu_o)\,\Omega \quad .$$

Laslett, Neil and Sessler[11] have computed the transverse fields arising from (4); they are such as to produce a force field

$$F = e(E - \beta B) = e^2 N \xi \left(U + \sqrt{\frac{i}{\omega}}\, V\right) e^{i(n\theta - \omega t)} \tag{5}$$

where U and V are nearly independent of n but depend on the dimensions of the beam and vacuum chamber; V is also proportional to the square root of the resistivity of the wall material. The sign of $\sqrt{i/\omega}$ is such that its real part is positive; thus the imaginary part has the same sign as ω. Usually $V \ll U$.

The force equation on the k^{th} particle then becomes

$$M\gamma\,(\ddot{z}_k + \nu_k^2\,\Omega^2\,z_k) = e^2(U + \sqrt{\tfrac{i}{\omega}}\;V)\,\sum_r z_r \tag{6}$$

Since we are considering a normal mode with frequency $\nu_o\Omega$, we have $\ddot{z}_k = -\nu_o^2\,\Omega^2\,z_k$. We divide by $(\nu_k^2 - \nu_o^2)$ and sum over all particles k:

$$M\gamma\,\Omega^2 \sum_k z_k = e^2(U + \sqrt{\tfrac{i}{\omega}}\;V)\,\sum_k \frac{1}{\nu_k^2 - \nu_o^2}\,\sum_r z_r \tag{7}$$

Replacing the sum by an integral over the distribution $f(\nu)$ of the intrinsic frequency of the particles (which may be taken as normalized to unity) we obtain the dispersion relation

$$\frac{M\gamma\,\Omega^2}{U + \sqrt{\frac{i}{\omega}}\;V} = Ne^2 \int \frac{f(\nu)\,d\nu}{\nu^2 - \nu_o^2} \tag{8}$$

We try to solve for a ν_o with positive real part. If the solution has a negative imaginary part, we have instability; a positive imaginary part gives stability.

It can be shown now that instability occurs only if ω is positive (i.e. $n > \nu_o$), and if N exceeds a certain threshold which depends on the width of the distribution $f(\nu)$ and on U, but only very weakly on V. On the other hand the rate of growth, when the threshold is exceeded, is proportional to V.

In the case of a bunched beam, many different Fourier components n are present. For a sharply bunched beam one can carry out an analysis similar to the above, and it is found that the term corresponding to $U + \sqrt{i/\omega}\;V$ is the sum of the terms from the individual Fourier components. The sign of the imaginary part then depends on whether ν_1 [the center of the distribution $f(\nu)$] lies in the lower or upper half of the interval between two integers. Instability can then arise only in the latter case, i.e. when

$$k + \tfrac{1}{2} < \nu_1 < k$$

where k is some integer.

This conclusion has not yet been tested experimentally, since all accelerators where this effect has been observed happen to have ν_1 lying just below the nearest integer. The CERN proton synchrotron ($\nu_1 \approx 6\tfrac{1}{4}$) exhibits a transverse coherent instability but that one is attributed to interaction with ions in the residual gas,[13] for which V can be shown to have the opposite sign from the one for resistive wall effects.

IV. Instabilities in Colliding Beams

In the Stanford electron-electron storage rings, the transverse resistive instability has been observed, and has been suppressed by the addition of an octupole magnet which increases the effective spread of oscillation frequencies ν. However, it has been observed that, when a beam in one ring is made to collide with the beam in the other ring, the less intense beam will grow in vertical size, thus reducing the number of particle collisions. This growth takes place even if the less intense beam contains very few particles, provided the other one is intense enough.

At first sight, the explanation seems to be a straightforward extension of the space-charge detuning theory of Section II. The formula (1) for change of frequency has to be modified because the magnetic and electric effects add rather than cancel when two beams of particles going in opposite directions intersect (in the electron-electron case we have equal charges and opposite currents, both repelling; for electrons colliding with positrons opposite charges attract as do equal currents). The intersection takes place only in a small fraction of the circumference. Thus (1) has to be modified by multiplying by the factor

$$\frac{1 + \beta^2}{1 - \beta^2} \cdot F \tag{9}$$

where F is the fraction of the circumference in which the beams intersect. Using this approach, Amman and Ritson[14] computed limits on the beam intensity in storage rings.

However, the beam blowup is observed to occur at currents considerably below the Amman-Ritson limit. Therefore more subtle effects than linear resonances seem to play a part. The simplest explanation, suggested by Robinson,[15] is that nonlinear resonances are excited by the strongly nonlinear form of the field due to a thin ribbon beam. Since nonlinear resonances are more closely spaced than linear ones, smaller values of $\Delta\nu$ may be expected to lead to blowup.

This hypothesis was tested by a computer program. The intense ribbon beam is represented by an impulse dependent on z, as follows:

$$F(z) = \begin{cases} \mathbf{D} & z < b \\ \mathbf{D}\,z/b & -b < z < b \\ -\mathbf{D} & z < -b \end{cases} \tag{10}$$

where 2b is the height of the ribbon beam, and D a measure of its strength. A particle is followed for many revolutions as it undergoes the impulse (10) once per revolution and moves in a linear focusing field the rest of the time. At regular intervals the position of the particle in (z,z') phase space is graphed.

Figure 1 shows the plots of particles for three sets of initial conditions in a case where D is about one fourth as large as the value which would cause instability in the linear theory. It is seen that with small initial amplitudes the oscillations remain bounded in amplitude, although they do not exactly lie on a smooth closed curve in phase space. For the largest initial amplitude plotted (2.5 times the height of the ribbon) the amplitude grows with time, as evidenced by the points that lie at large values of z and z'. The growth proceeds without any discernible order, somewhat in the manner of oscillations excited by noise. This shows that if the initial amplitude is large enough — in this case 2.5 times the height of the ribbon — the motion is unstable in the long run. Whether it would also turn unstable eventually for smaller initial amplitudes (after very many revolutions) is not known; the lack of closed curves in phase space suggests this possibility. In any case, damping by synchrotron radiation (which was not included in the computations) will limit the oscillations to the amplitude reached in the characteristic damping time — about 0.1 seconds for the Stanford ring.

If the force (10) is replaced by an error function (corresponding to a beam whose charge distribution is Gaussian rather than uniform within sharp limits) we obtain the diagram of Fig. 2. Here the oscillations are undoubtedly stable, since the points lie on smooth closed curves — at least within plotting accuracy. Since the actual beam is likely to be more nearly Gaussian than uniform, it appears that we still do not have a complete picture of the beam blowup.

However, these calculations are based on a grossly simplified model. The computer program was modified to provide for motion in the horizontal as well as vertical direction, with the vertical force depending on horizontal position and vice versa. Furthermore, the exciting ribbon beam was given small coherent oscillations of its own. In this case (which is not as easily graphed) it was found that beam blowup occurred in many cases, particularly when the frequency of coherent oscillations of the strong beam was close to the resonant frequency of the particles in the blown-up beam. Here again the details of the blowup resembled what one would expect from noise excitation, but now the beam was unstable whether the exciting beam was taken to be uniform or Gaussian in form.

The coherent transverse resistive instability is also enhanced by the factor (9), as pointed out by Sessler.[16] He shows that this instability can arise from the mutual interaction of two intense beams, and is most dangerous when their intrinsic frequencies of oscillations are close together.

Successful operation of the Stanford electron storage rings has now been achieved, as will be reported at this conference by B. Gittelman,[17] by making the ν-values of the two rings different to inhibit the two-beam coherent instability; increasing the vertical size of the beams slightly so as to reduce the Amman-Ritson detuning effect and the nonlinear blowup, and inserting nonlinear lenses to inhibit the coherent single-beam instability. Extrapolation of present theories indicates that the projected 3-BeV electron-positron storage rings should work stably at the design intensity of about 1 ampere circulating per beam.

One may ask how these results apply to large proton storage rings such as the ones now being projected at CERN. Here the Amman-Ritson detuning is very much smaller than for the electron or positron rings ($\Delta\nu \leqslant 0.001$), both because of the higher energy and because the beam cross sections are larger. However, in the electron rings damping from synchrotron radiation helps limit the amplitudes; in the proton rings this effect is absent, and stability has to be maintained for hours rather than seconds. Computations are practical for at most of the order of one million beam intersections, corresponding to about one second. Such computations were performed; they show the amplitude increasing by less than one part in a thousand in one second due to the nonlinear resonance effect. This makes it very likely that no appreciable blowup due to this cause will occur even for very long storage times.

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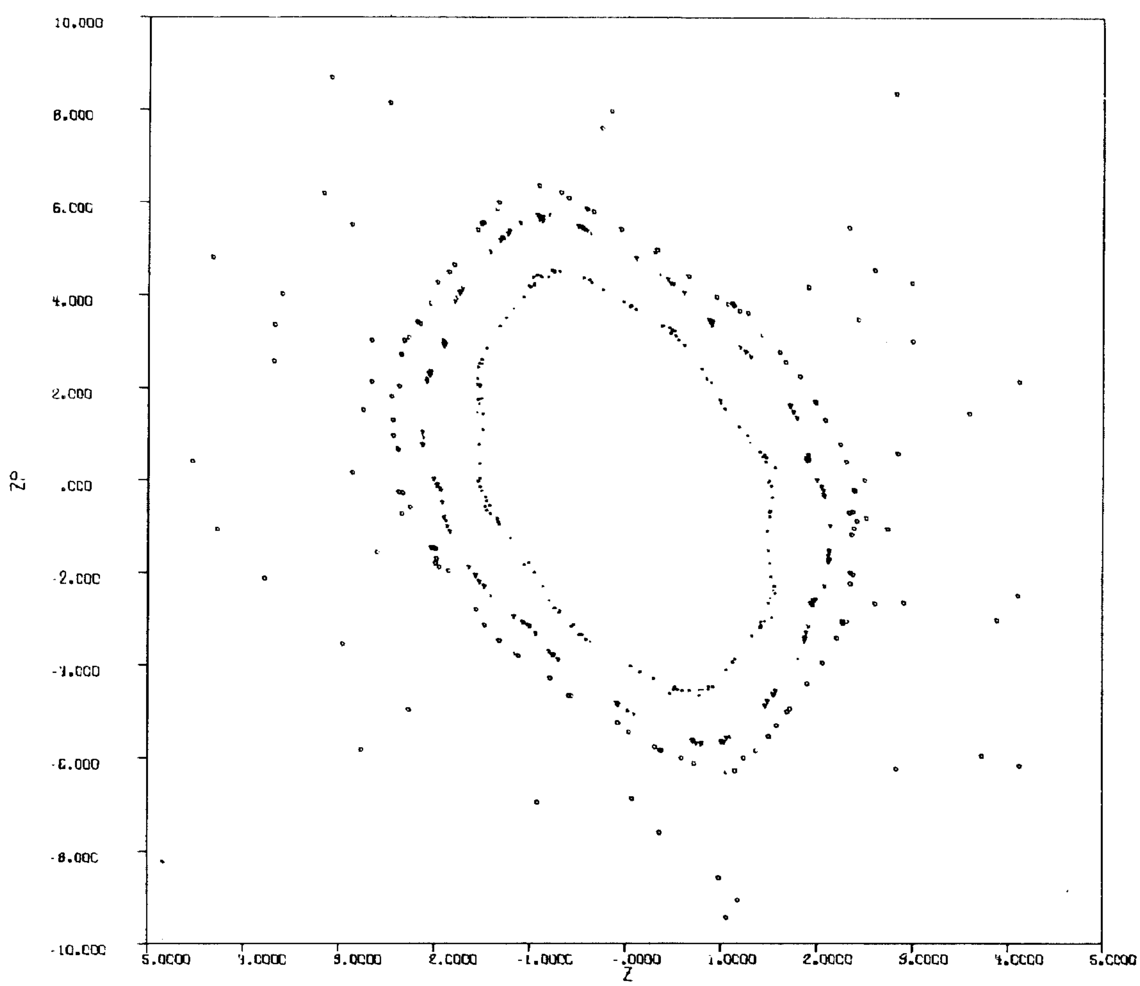

Figure 1. Phase-space diagram for particles traversing beam with sharp boundaries at z = ± 1, followed for 2000 turns. Dots — initial amplitude 1.5; triangles — initial amplitude 2.0; squares — initial amplitude 2.5.

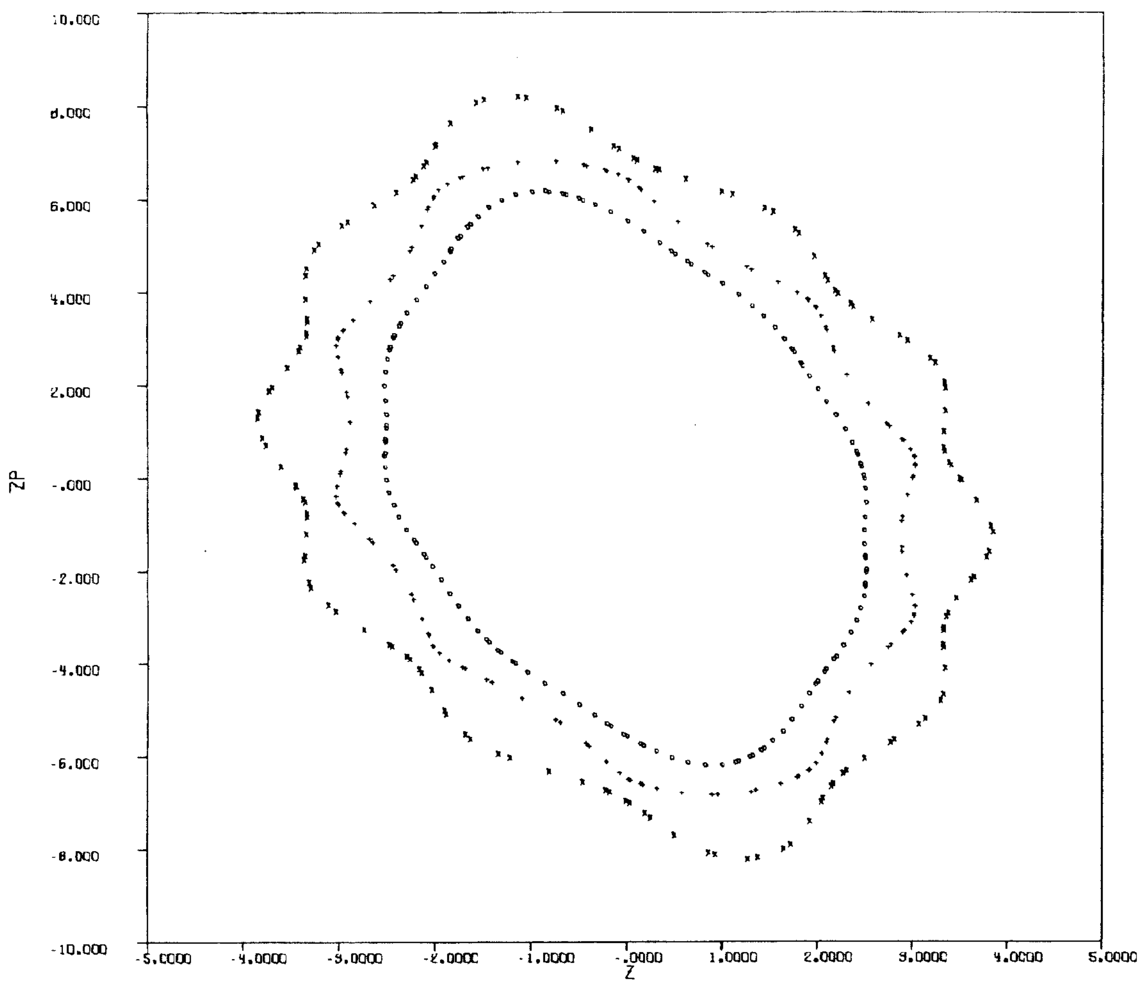

Figure 2. Same as Fig. 1, but with beam of Gaussian shape, same intensity and central density as in Fig. 1. Squares — initial amplitude 2.5; +'s — initial amplitude 3.0; X's — initial amplitude 3.5.