

ANALYSIS OF RESONANT RADIO-FREQUENCY COUPLING SYSTEMS

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Summary

This paper considers resonant coupling circuits useful in coupling RF energy to cyclotron RF resonators. Two specific circuits are considered. In the first circuit, the anode of the RF power tube is connected directly to the coupling loop. In the second, a length of transmission line connects the anode to the coupling loop. Inductive (loop) coupling is assumed for the sake of this presentation. With slight modification, the expressions derived apply equally well to capacitively coupled circuits.

Simple Coupling Circuit

The first type of coupling circuit to be considered is that in which the coupling loop is connected directly to the anode of the power tube. This circuit is shown in Figure 1. The fact that the power tube is shown in the grounded grid connection is not essential to the following discussion. It may be shown that the coupling loop and the resonator referred to terminals A-B of the coupling loop may be represented by the equivalent circuit of Figure 2. In this circuit, k is the coupling coefficient of the loop and is given by

$$K = \frac{V_a}{V_{dee}} \quad \left| \quad I_1 = 0 \right.$$

L<sub>dee</sub>, R<sub>dee</sub>, and C<sub>dee</sub> are the lumped circuit parameters of the resonator referred to the dee. These parameters may be determined from the Q of the resonator and the RF driving power required to produce a given value of dee voltage. C<sub>a</sub> is the total capacity from anode to ground including the inter-electrode capacitance of the power tube as well as any externally added anode to ground capacity. V<sub>a</sub> is the anode to ground voltage. L<sub>l</sub> is the self-inductance of the coupling loop. R<sub>l</sub> is the resistance of the coupling circuit which gives rise to energy loss in the coupling circuit.

Zero Power Transfer

Let us first assume that the coupling circuit is lossless and transmits no power to the resonator. This assumption leads to a particularly simple relationship between dee and anode voltage which, under certain conditions, is not a bad approximation. This simple model also gives insight into the two modes in which the coupling circuit may operate. After discussing the simple model, the more general case will be discussed.

If the coupling circuit is lossless and there is no power being transmitted through it

the equivalent circuit of Figure 2 may be simplified to the one shown in Figure 3. X<sub>Ca</sub> is the capacitive reactance of the anode capacity, X<sub>Ll</sub> is the inductive reactance of the coupling loop, and X<sub>dee</sub> is the reactance of the resonator referred to the coupling loop. This circuit has two natural frequencies or modes, one due to the resonator and the other due to the loop inductance and the anode capacitance. Since in practical circuits the resonator and coupling circuit are not tuned close together and since the resonator has a very high Q, the resonator mode will be very close to the natural frequency of the resonator and the coupling loop mode will be very nearly the natural frequency of the coupling loop and anode capacity. If this coupling circuit is to be useful, it must operate in the resonator mode in order that kV<sub>dee</sub> will not be vanishingly small. If the circuit is operating in the resonator mode, then we have

$$jX_{dee} = -jX_{Ca} - jX_{Ll}$$

$$\frac{V_a}{kV_{dee}} = - \frac{-jX_{Ca}}{jX_{dee}}$$

Combining these expressions one obtains

$$\frac{V_a}{kV_{dee}} = \frac{1}{1 - \frac{X_{Ll}}{X_{Ca}}} \approx \frac{1}{1 - \left(\frac{f_r}{f_l}\right)^2} \quad (1)$$

Where f<sub>r</sub> is the natural frequency of the resonator and f<sub>l</sub>, the natural frequency of the coupling circuit, is given by

$$f_l = \frac{1}{2\pi \sqrt{L_l C_a}}$$

Equation 1 shows that the ratio of dee to anode voltage depends only on the natural frequency of the resonator and the natural frequency of the coupling circuit. If the coupling circuit is tuned above the resonator V<sub>a</sub> and kV<sub>dee</sub> are in phase. If the coupling circuit is tuned below the resonator, V<sub>a</sub> and kV<sub>dee</sub> are 180° out of phase.

In order to check the validity of Equation 1, a loop was placed in a resonator and the ratio of V<sub>a</sub> to kV<sub>dee</sub> measured for four values of capacity C<sub>a</sub> loading the loop. The natural frequencies f<sub>r</sub> and f<sub>l</sub> were also measured and substituted in Equation 1 to compute the ratio of V<sub>a</sub> to kV<sub>dee</sub>. The measured and computed results are presented in Table 1.

TABLE I  
Summary of Coupling Circuit Test

Ca	f1	fr	$\frac{V_a}{kV_{dee}}$ (calc.)	$\left  \frac{V_a}{kV_{dee}} \right $ (meas.)
500 pf	10.1	14.2	-1.0	1.06
275 pf	13.0	14.2	-5.2	4.0
200 pf	15.9	14.2	+4.9	3.52
100 pf	22.5	14.2	+1.66	1.60

Table I indicates that Equation 1 is valid if the ratio of  $V_a$  to  $kV_{dee}$  does not become too large. The discrepancy between observed and calculated values is due to the assumption that the circuit is lossless and transmits no power.

To design a coupling circuit using Equation 1, the self-inductance of the loop  $L_1$  must be known.  $L_1$  depends on the size of the coupling loop which in turn depends on the desired value of coupling coefficient  $k$ . Once the size of the loop is fixed, its inductance may be calculated. If the loop is longer than  $1/12$  of the electromagnetic wave length, the current in the loop becomes nonuniform and the inductance becomes a function of frequency becoming very large as the loop approaches a quarter wave in length. If, however, the loop is less than  $1/12$  wave length long, the current distribution is approximately uniform and the loop inductance is nearly constant and given by

$$L_1 = .0051 \left[ 2b \ln \left( \frac{c-a}{a} \right) + c \ln \left( \frac{2b-a}{a} \right) \right] \mu h \quad (2)$$

where  $a$ ,  $b$ , and  $c$  are in inches and are given in Figure 4.

A loop with  $a = 1''$ ,  $b = 5.5''$  and  $c = 16''$  was constructed and resonated with a 75.2 pf capacitor at 32.7 MC. The loop inductance is therefore .314 microhenries. Computing  $L_1$  from the loop dimensions and Equation 1 gives .338 microhenries. For this case the loop was approximately 27 electrical degrees long.

#### Finite Power Transfer

As was observed in Table 1, the approximation leading to 1 is not accurate if the ratio  $V_a$  to  $kV_{dee}$  is large. If power is transmitted through the coupling system but losses in the coupling system are neglected, the equivalent circuit is as illustrated in Figure 5a.

The phasor diagram associated with this circuit is shown in Figure 5b.  $I_1$  the peak RF current flowing in the loop has two phase quadrature components  $I_i$  and  $I_r$ .  $I_i$  is in phase quadrature with  $V_a$  and represents the current flowing in  $C_a$ .  $I_r$  is in phase with  $V_a$  and  $I_r V_a$  represents power being transferred to the resonator.  $I_i$  and  $I_r$  are given by

$$I_i = \frac{V_a}{X_{Ca}}$$

$$I_r = \frac{2Pt}{V_a}$$

where  $P_t$  is the power being transferred to the resonator. The phase angle  $\alpha$  between  $V_a$  and  $I_1$  is then given by

$$\alpha = \tan^{-1} \left( \frac{2Pt X_{Ca}}{V_a^2} \right)$$

the modulus of  $I_1$  is then

$$|I_1| = \left[ \left| \frac{V_a}{X_{Ca}} \right|^2 + \left| \frac{2Pt}{V_a} \right|^2 \right]^{1/2}$$

and multiplying by  $X_{L1}$   $V_1$  is then

$$|V_1| = |V_a| \left[ \left| \frac{X_{L1}}{X_{Ca}} \right|^2 + \left| \frac{2Pt X_{L1}}{V_a^2} \right|^2 \right]^{1/2}$$

and

$$|kV_{dee}| = \left[ |V_a|^2 + |V_1|^2 - 2|V_a||V_1|\cos\alpha \right]^{1/2} \quad (3)$$

The phase shift  $\beta$  between  $V_a$  and  $kV_{dee}$  is

$$\beta = \sin^{-1} \left[ \frac{V_a}{kV_{dee}} \left[ \left( \frac{X_{L1}}{X_{Ca}} \right)^2 + \left( \frac{2Pt X_{L1}}{V_a^2} \right)^2 \right]^{1/2} \sin\alpha \right] \quad (4)$$

Equations 3 and 4 describe the relationship between  $V_a$  and  $kV_{dee}$  more accurately than Equation 1. Unlike Equation 1 the phase shift between  $V_a$  and  $kV_{dee}$  is always finite if  $P_t$  is greater than zero. As  $P_t$  approaches zero  $kV_{dee}$  approaches  $|V_a| - |V_1|$  which leads to the result of Equation 1.

#### Phase Shift

If the power tube is to be operated as a self-excited oscillator which takes its drive from the resonator then the phase shift is particularly important since the relationship between the anode voltage and drive voltage is critical. Examination of Figure 5b shows that the phase shift  $\beta$  depends on both  $\alpha$  and the ratio of  $V_a$  to anode voltage. If  $|V_a/kV_{dee}| < 1$  then the phase shift depends primarily on  $\alpha$ . For practical self-excited oscillators  $\alpha$  must be small; therefore,  $\alpha$  may be approximated by

$$\alpha \approx \frac{2Pt X_{Ca}}{V_a^2} = \frac{I_r}{I_i}$$

Thus if the phase shift  $\beta$  is to be kept close to  $0^\circ$  or  $180^\circ$  the ratio of resistive to reactive current at the anode must be kept small.

In the design of a resonant coupling circuit for a self-excited oscillator it is desirable to maintain the ratio  $I_i/I_r$  between 7 and 10.<sup>1</sup> If the ratio falls below 7, the phase shift becomes excessive. If the ratio exceeds 10

losses in the coupling circuit due to high circulating currents become excessive.

Losses in the coupling circuit are given by  $I_1 R_1 / 2$  provided that  $R_1$  is very much less than  $X_{L1}$ . It is of course necessary to make the perimeter of the coupling loop sufficiently large to keep these losses down.

Resonant Transmission Line Coupling

If the frequency of the resonator changes, the simple coupling circuit described above must be retuned to maintain a constant ratio of dee to anode voltage. Resonant transmission line coupling systems allow varying the resonator frequency over more than a two to one range without retuning. Such systems also make it possible to separate the power tube from the resonator.

Zero Power Transfer

Let us start by assuming a lossless transmission line with no power transfer. As before, this leads to an approximation which is useful in many cases. If there is no power transfer, the transmission line is terminated on one end by the anode capacity  $C_a$  and on the other by the resonator and loop reactance. This is illustrated in Figure 6. Again we have two coupled resonant circuits and two modes. Since in practical circuits the two modes are not tuned close together and since the resonator is high Q, the resonant frequency of the resonator mode is almost exactly the same as the natural frequency of the resonator,  $f_r$ . From the equations for a resonant transmission line we have

$$\frac{V_a}{V_b} = \frac{\sin \gamma l_1}{\sin \gamma l_2} \tag{5}$$

Where  $\gamma$ , the propagation constant of the TEM transmission line mode, is given by

$$\gamma = \frac{2\pi f}{\sqrt{\epsilon_r} c}$$

$c$  is the velocity of light and  $\epsilon_r$  is the relative di-electric constant of the transmission line.  $l_1$  is computed from the relationship

$$Z_0 \tan \gamma l_1 = X_{C_a}$$

$Z_0$  is the characteristic impedance of the transmission line. From a derivation similar to that for Equation 1 we obtain

$$\frac{V_b}{kV_{dee}} = \frac{1}{1 + \frac{X_{L1}}{Z_0 \tan \gamma l_2}}$$

Combining with 5 this becomes

$$\frac{V_a}{kV_{dee}} = \frac{\sin \gamma l_1}{\sin \gamma l_2 + X_{L1} \cos \gamma l_2} \tag{6}$$

Equation 6 gives the ratio of dee to anode voltage if the anode capacity  $C_a$ , the loop inductance  $L_1$ , the line length  $L$  and the resonator frequency  $f_r$  are given. Unlike the simple circuit, it is not possible to establish the ratio  $V_a/kV_{dee}$  as a function of only the two natural frequencies of the system. However, given line length  $L$  anode capacity  $C_a$  and the two natural frequencies of the system, one may calculate  $X_{L1}$  and evaluate Equation 6. Equation 6 is similar to Equation 1 in that  $V_a/kV_{dee}$  may become infinite and also in that the phase shift from anode to resonator is either 0 or 180°.

A check on the validity of Equation 6 was made by connecting a capacity loaded transmission line to a loop in a resonant cavity. The results of this test are summarized in Table II.

TABLE II

Computed vs. Calculated Values of  $V_a/kV_{dee}$  for Transmission Line Coupling

$C_a$	$f_r$	$\gamma L$	$L_1$	$\frac{V_a}{kV_{dee}}$ (calc.)	$\frac{V_a}{kV_{dee}}$ (meas.)
500 pf	26.8	79°	.5μh	.10	.10
200 pf	26.8	79°	.5μh	.23	.21
100 pf	26.8	79°	.5μh	.40	.38
200 pf	14.2	42°	.5μh	2.33	2.50
100 pf	14.2	42°	.5μh	.79	.80

Finite Power Transfer

The more general case in which power is transferred through the circuit is given an excellent treatment by K. R. MacKenzie.<sup>1</sup> This approach consists of a graphical analysis of the transmission line in which the magnitude and phase of the forward and backward waves on the line are computed by applying the boundary conditions on the line. This method provides a great deal of physical insight.

Phase Shift

The phase shift  $\beta$  between  $V_a$  and  $V_b$  is given by

$$\beta = \gamma L + \tan^{-1} \left[ \frac{1}{\frac{\tan \gamma l_1 - 2 Z_0 \tan \gamma l_1 P_t}{V_a^2}} \right] + \tan^{-1} \left[ \frac{1}{\frac{\tan \gamma l_2 - 2 Z_0 \tan \gamma l_2 P_t}{V_b^2}} \right] \tag{7}$$

Equation 7 is derived from K. R. MacKenzie's phasor diagram analysis. If  $P_t = 0$  then the phase shift  $\beta$  is either 0 or 180° depending on whether  $L$  is less than or greater than  $l_1$ . The

term

$$\frac{2Z_0 \tan \gamma l Pt}{V^2} = \frac{I_{\text{resistive}}}{I_{\text{reactive}}}$$

introduces phase shift. Thus the ratio of resistance to reaction currents at each end of the line determines the phase shift in the line. In order to calculate the total phase shift between  $V_a$  and  $kV_{dee}$  one must combine the phase shifts given by Equations 4 and 7.

If phase shifts are to be kept low without excessive losses in the coupling circuit ( $I_{\text{reactive}}/I_{\text{resistive}}$ ) should be in the range 7-10.<sup>1</sup>

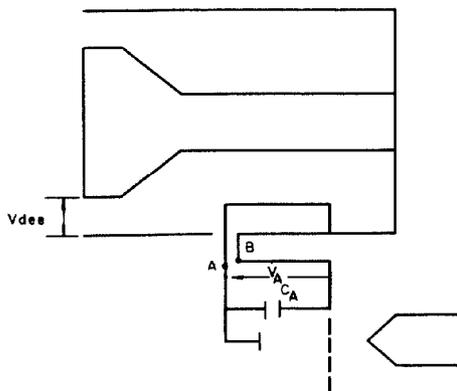


Fig. 1. Resonator and Coupling Loop.

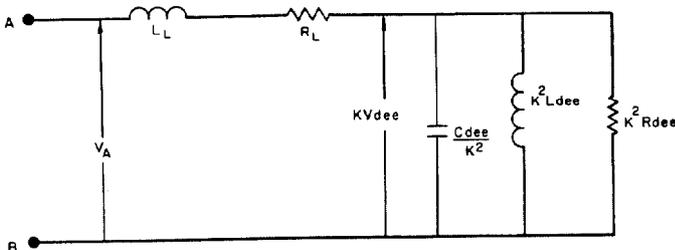


Fig. 2. Equivalent Circuit of Coupling Loop and Resonator.

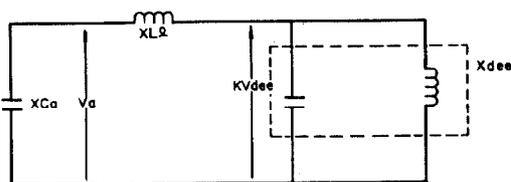


Fig. 3. Equivalent Circuit for Zero Power Transfer.

Conclusion

In starting the design of a resonant coupling system it is useful to use the approximate coupling formulas Equations 1 or 6. In this way one may gain an insight into the system without being burdened by long cumbersome expressions. The more exact solutions are useful once a tentative coupling system has been established.

References

1. K. R. MacKenzie, Notes on Coupled Circuits from Lectures by K. R. MacKenzie, UCLA, 1954.
2. Ramo, S. and Whinnery, J. R., "Fields and Waves in Modern Radio", John Wiley and Sons, New York, 1953.

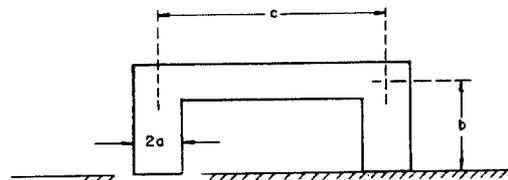


Fig. 4. Loop Dimensions for Use in Equation 4.

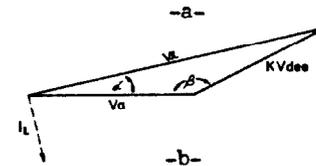
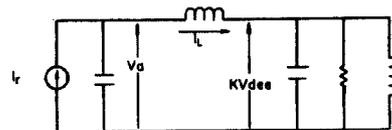


Fig. 5a. Equivalent Circuit for Finite Power Transfer and Fig. 5b. Phasor Diagram.

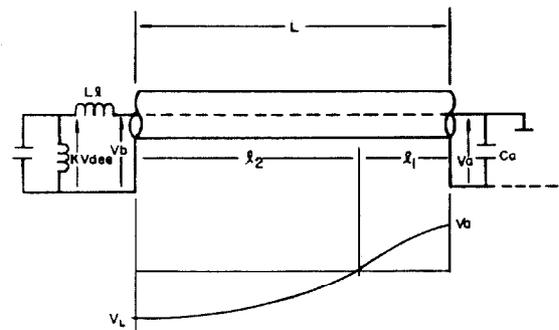


Fig. 6. Resonator Transmission Line and Voltage Standing Wave.