A SELF-CONSISTENT MODEL FOR EMITTANCE GROWTH OF MISMATCHED CHARGED PARTICLE BEAMS IN LINEAR ACCELERATORS*

R. P. Nunes#, F. B. Rizzato†, R. Pakter‡
Instituto de Física, UFRGS, Av. Bento Gonçalves 9500, C.P. 15051, Porto Alegre, Brasil.

Abstract

This work analyzes the envelope dynamics of a high-intensity charged particle beam focused by constant magnetic field. As known, beams with mismatched envelopes decay into its equilibrium state with a simultaneous increasing of its emittance. This implies that in the stationary regime the beam transverse phase-space is characterized by a tenuous population of high velocity particles surrounding a dense population of low velocity particles. To describe this emittance growth, a test-particle approach has been employed by us to develop a simplified self-consistent macroscopic model. The model has been compared with full *N*-particle Gauss simulations and the agreement is shown to be quite reasonable.

INTRODUCTION

Magnetically focused beams of charged particles can relax from a nonstationary into a stationary state through emittance growth [1]. This is the case of beams with initially mismatched envelopes, flowing along the symmetry axis of focusing systems with a constant magnetic field. Gluckstern has shown [2] that the physical mechanism that governs this emittance growth is large resonant islands [3] beyond the beam border. The formation of these resonant islands is induced by the initial mismatch of the beam, which can drive particles residing in the beam vicinity to excursions with high amplitude. The resonant coupling above progressively converts energy of the beam macroscopic oscillation into kinetic energy to the microscopic chaotic movement of the outer particles, causing the decay of the beam envelope and, as a consequence, the increasing of its emittance. This process continues until the equilibrium state of the system is reached, when emittance saturates. In this situation, the beam can be described by two distinct populations of particles: a dense one, formed by low velocity particles which compose the beam core, and a tenuous one, surrounding the previous population and formed by high velocity particles, which compose the beam halo.

The splitting of the initial beam into a cold dense and a hot tenuous population suggests describing the latter by test particles. The test particles will interact with the beam by the means of the Gluckstern's resonances explained above. The information carried by the test particles has been used by us to model the dynamics of the emittance growth as the beam axially evolves in the focusing channel. In our model, the importance of the reaction of the test particle population over the beam core has been shown to depend only of the amount N_h of halo particles that compose the whole N-particle beam, given here by the fraction $f=N_h/N$. This fraction f has been obtained considering the initial and the final (stationary) beam densities with the aid of equations for the conserved quantities of the entire system, formed by the beam particles and fields.

THE DEVELOPED MODEL

The system modelled here is formed by an azimuthally symmetric beam of charged particles, moving inside a circular conducting pipe and focused by a constant solenoidal magnetic field, aligned with the pipe axis. The beam is initially cold, which means that its initial emittance can be neglected. Since space-charge beams are fairly homogeneous [1], the beam cross section initially obeys a uniform profile

$$n(r) = \begin{cases} N / \pi r_o^2 & \text{if } r \leq r_o \\ 0 & \text{if } r > r_o \end{cases} \qquad (1)$$

r is the radial variable measured from the symmetry axis and r_o denotes the initial value for the beam radius. As the beam evolves, particles are expelled from the beam core and start to populate an extended hot halo. At this point Eq. (1) is no longer valid and it is necessary to impose a model for the density of halo particles. Considering that our interest here is to relate beam quantities of its initial state with quantities of its stationary state, a simple approach is just to model the asymptotic beam halo density, allowing particles of the beam core still follow an uniform density as in Eq. (1), with its radius now specified by $r_c<r_o$.

Invoking our previous assumption that the halo can really be modelled by test particles, in Figure 1(a) it is presented by us the transverse phase-space for test particles initially distributed in a region $I=[r_o, r_o(1+\delta)]$ at the beam border, with $r_o=1.6$ and governed by the so called particle-core model equations [7]. The equilibrium beam radius is $r_{eq}=1$ and $\delta<<1$. In Figure 1(b), it is shown the phase-space of a full self-consistent simulation, based on Gauss' law [5][8] for a number up to N=20000 and after the beam reaches its stationary regime, which numerically means 150 beam envelope cycles. The direct comparison between the last two figures assures a nice correspondence. The beam density has been taken as a dense population along the horizontal branch (beam core particles), added by a tenuous population lying in the phase-space separatrix (beam halo particles).

*CNPq and FAPERGS, Brasil. AFOSR FA9550-06-1-0345, USA.
#rogerpn@if.ufrgs.br, †rizzato@if.ufrgs.br, ‡pakter@if.ufrgs.br

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Considering the semicircular shape of the separatrix, the linear density of the halo particles can be expressed by

$$n_{halo}(r) = 2\int_{r_i'}^{r_f'} \sigma(r,r')dr' \approx \frac{2fN}{\pi\sqrt{r_h^2 - r^2}} \qquad (2)$$

in which r_h is phase-space radius of the separatrix, r_i' and r_f' delimits the semicircular branch of the separatrix. More details can be found in reference [8]. For a quantitative test, the equation above has been compared with the histogram of the halo particles identified in the phase-space of full simulation. The comparison shows to be relatively good for $r_h \approx 2$.

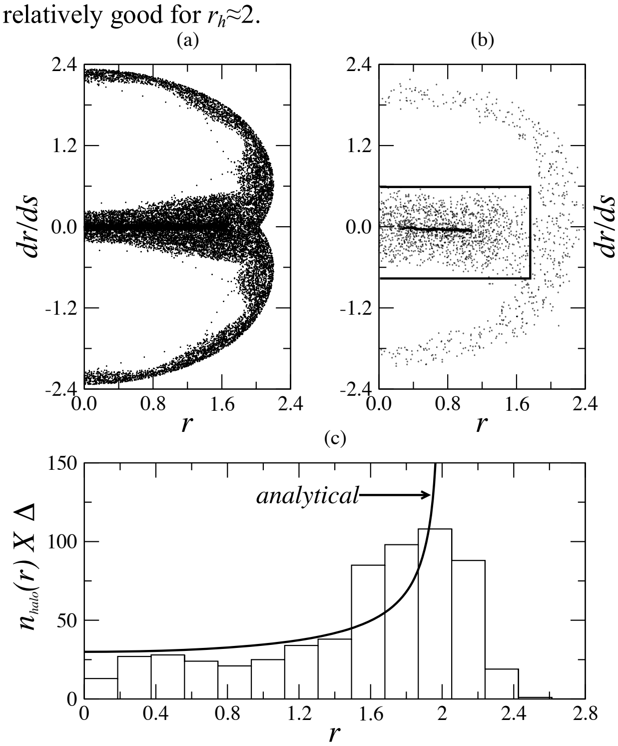

Figure 1: Phase-spaces for (a) test particles, and (b) full simulations. Panel (c) displays the histogram of halo particles from panel (b) (with bin size of Δ=0.18) in comparison with our model for the halo density.

Notwithstanding the complexity of the dynamics connecting the beam initial state to its stationary state, an exact set of governing equations can still be obtained for some quantities of the beam for all of its dynamics. One of them is for the beam envelope r_b, which is related to the root mean square (rms) beam radius $<x^2+y^2>^{1/2}$ through

$$r_b^2 \equiv 2\langle x^2 + y^2 \rangle \qquad (3)$$

in which the brackets denote particle average or, equivalently, phase-space average, the definition of which will be made more precise and operational later. The envelope equation itself reads [6]

$$\frac{d^2 r_b(s)}{ds^2} = -r_b(s) + \frac{1}{r_b(s)} + \frac{\varepsilon^2(s)}{r_b^3(s)} \qquad (4)$$

in which $\varepsilon(s)$ is the beam emittance that can depends of the axial distance s. Beam emittance is defined in the form

$$\varepsilon^2 = 4\left(\langle x^2+y^2\rangle + \langle x'^2+y'^2\rangle - \frac{\langle x^2+y^2\rangle'^2}{4}\right) \qquad (5)$$

in which the primes indicate derivatives with respect to the axial distance s. Equation (5) is only defining, but it can be associated with the energy conserving relation

$$\frac{1}{2}\langle x'^2+y'^2\rangle + \frac{1}{2}\langle x^2+y^2\rangle + \xi(s) = const. \qquad (6)$$

to provide information about the emittance growth process. In Eq. (6), $\xi(s)$ is the average self-field energy per particle defined by

$$\xi(s) = \frac{1}{4\pi}\int \left|\nabla_\perp \psi\right|^2 d^2r \qquad (7)$$

and ψ is the dimensionless scalar electromagnetic potential governed by the Poisson equation

$$\nabla_\perp^2 \psi = -\frac{2\pi}{N} n(\mathbf{r}, s) \qquad (8)$$

Given the nice agreement of the developed model for the halo particle density in the beam stationary state, it is possible to decompose the beam radius r_b in contributions from the beam halo, given by its envelope r_h, and contributions from the beam core, given by its envelope r_c. In order to do so, it will be necessary to break up all the statistic summations into two pieces: one coming from the cold particles of the beam core and other coming from hot particles of the beam halo. Specifically, based on the phase-spaces shown in Figure 1 (a) and (b), ensemble averages of radial quantities can be calculated as follow

$$\langle x^2+y^2\rangle = \sum_{n=1}^{N} \frac{(x^2+y^2)_n}{N} \qquad (9)$$

in which n indexes each particle of a total of N that composes the beam phase-space. Splitting the summation above over particles originating from the beam halo and the beam core

$$\langle x^2+y^2\rangle = \frac{N_c}{N}\sum_{\text{core}} (x^2+y^2)_n + \frac{N_h}{N}\sum_{\text{halo}} (x^2+y^2)_n \qquad (10)$$

in which N_c is the number of particles that remain inside the beam core in its stationary state. The conservation of beam particles $N=N_c+N_h$ have been kept.

Observing our previous definition for the fraction f of particles pertaining to the beam halo and for the beam envelope expressed by Eq. (3), it is possible to rewrite Eq. (10) in a more convenient way

$$r_b^2 = 2\langle r^2\rangle = 2(1-f)\langle r^2\rangle_c + 2f\langle r^2\rangle_h \qquad (11)$$

Since it has been considered that the beam has a uniform density core, the first summation of Eq. (10) is easy to carry out for any axial distance s. The same does not occur for the second summation, in which the dynamics of the halo distribution must be known. However, in the asymptotic limit $s \to \infty$, this summation can be performed using our developed model for the halo density, described by Eq. (2). After straightforward algebra, it is possible to show the important relation for the beam stationary state [4] [8]

$$r_b^2(s \to \infty) = (1-f)r_c^2 + f r_h^2 \qquad (12)$$

If one now supplements Eqs. (4)-(8) with the initial condition at beam entrance $r_b(s=0)=r_o$ augmented by a condition of straight injection $r_b'(s=0)=0$, estimates on the stationary state $s\to\infty$ as a function of the beam radius r_o of an initially mismatched beam become possible. In this procedure, it must be considered the asymptotic phase-space quantities that has been previously defined, namely the separatrix radius r_h and the beam core radius r_c. These quantities are necessary to evaluate radial integrations over Eqs. (7) and (8). Also, it is necessary to made use of the information provided by Eq. (12). Established the equations for the initial and the stationary state of the beam and solved the system formed by them, a polynomial of degree 2 in f has been found, whose solution can be numerically calculated. For $r_h\approx 2$, $r_c\approx 1$, and $r_o=1.6$, the value of the fraction is $f=0.102$, which is very close to the value provided by full simulations $f_{\text{simul}}\approx 0.091$.

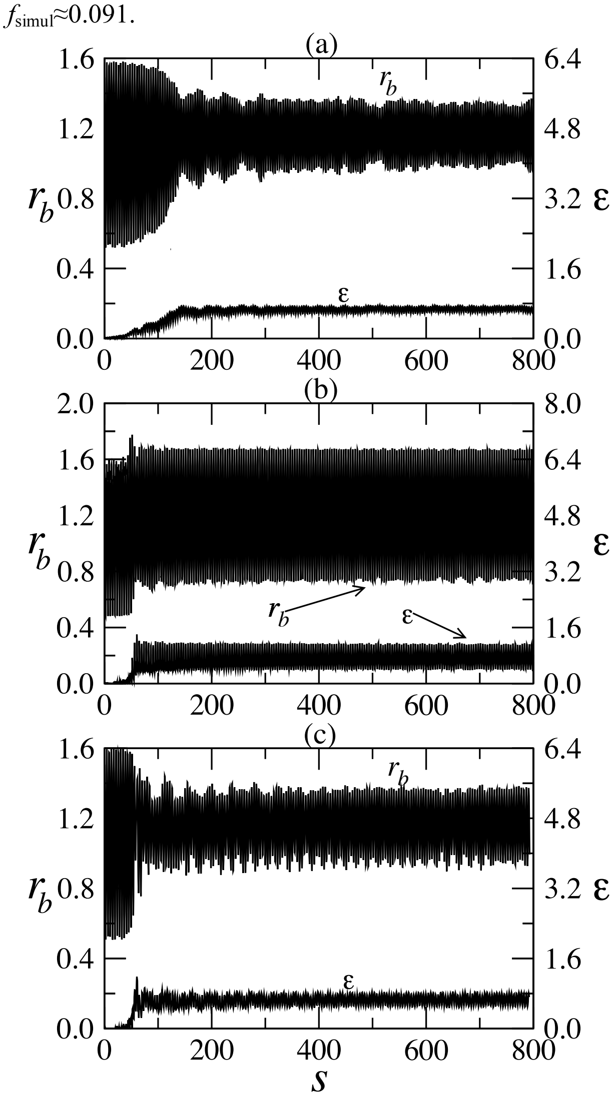

Figure 2: Emittance $\varepsilon(s)$ growth and envelope $r_b(s)$ dynamics in: (a) full simulations, (b) the case where the core $r_c(s)$ oscillates with fixed amplitude, and (c) in the self-consistent simplified model.

BEAM DECAY DYNAMICS

Once beam quantities corresponding to the stationary state have been predicted, information about its dynamics toward its equilibrium state can also be obtained. The overall emittance for the beam is defined in Eq. (5). As has been done earlier, the summation of the velocity term in Eq. (5) can be divided into its core and in its halo components. The resulting expression takes the form

$$\varepsilon^2 = r_b^2\left[(1-f)r_c'^2 + 2f\left\langle r'^2\right\rangle_h - r_b'^2\right] \qquad (13)$$

in which the average subscripted with h now denotes average only over the test particle population, which is the one modelling the halo.

Our results for emittance growth are summarized by comparisons of Figure 2 (a) with Figure 2 (b) and (c). In panel (a), it has been plotted the results of full Gauss simulations. In panel (b), it has been plotted the results for the rms envelope radius r_b of our model. In this case, the beam core envelope r_c is governed by Eq. (3) with ε set to zero. It is noticed that as soon as phase-mixing takes place due to the chaotic dynamics of the ejected test particles, emittance ceases to grow even though the core continues its oscillatory dynamics. Emittance exhibits much larger oscillations around the average asymptotic value since there is no decay of the beam envelope. The observed envelope growth is clearly a wrong result which has been fixed in panel (c). For that, the observed emittance growth is feedbacked in Eq. (4), forcing the beam envelope to decay, improving its dynamical description as shown.

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