CALCULATING THE NONLINEAR TUNE SHIFTS WITH AMPLITUDE USING MEASURED BPM DATA

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Abstract

An algorithm is proposed to calculate the approximate tune shifts with amplitude using only the linear transfer map of a circular accelerator and with little or no information on higher order nonlinearities. To extract information about the nonlinear dynamics, the decay rate of the average amplitude of the particle distribution after an instantaneous transversal kick is used. This method works when strong, low-order resonances are not present, that is where the linear lattice rather than the nonlinear driving terms dominates the machine dynamics. Nonlinear normal form transformation and differential algebra methods are employed to establish the connection between measurement results and the nonlinear tune shifts with amplitude. Proposed algorithm is applicable to a wide range of circular accelerators.

INTRODUCTION

Finding the nonlinear tune shift depending on the position of the particle in the beam might not be an easy task, because the nonlinear component of the dynamics is not known to the desired precision.

There is still a way to approximate the tune shift, if there is a set of specific measurements and some extra information which is usually available: about the geometry of the beam and the linear optics effect on the particles (in the form of the one-turn linear transfer matrix).

Consider the problem of evaluation of the tune shift with amplitude in the nonlinear case using some extra information obtained by the specific kind of measurements. All the proposed methods have been tested on the Tevatron model [1] and measurements [2], but the algorithm for finding the tune shift with amplitude stays valid for any other synchrotron, as long as one can proceed with a linear normal form transformation. The normal form transformation [3] is at the core of the method.

Suppose that one only has the information on the linear component of the dynamics of the particles in the accelerator. Assume that there is some extra information available: the size of the beam, the particle distribution type and also the results of the special type of measurements of the beam position. The corrector is introduced into the accelerator optics to kick the beam in the horizontal or vertical direction. Once the strength of the corrector is known, the displacement of the center of the beam can be found. After the corrector is turned on and off instantaneously, the amplitude of the beam center of mass decreases due to the filamentation of the beam, not the damping, as the motion is symplectic. The position of the center of mass of the beam is then registered after each turn of the particles. One sample of the measurement data for the horizontal position is shown in Fig. 1.

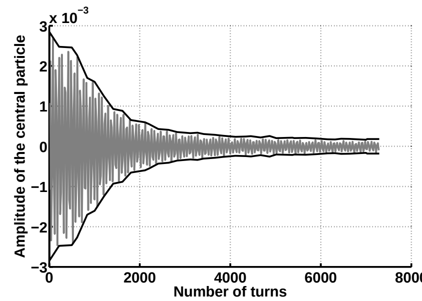

Figure 1: Measurement results: horizontal position of the center of mass over a number of turns and its envelope.

The normal form transformation yields that in the nonlinear case the tune can be represented in the following form:

$$\mu = \mu_0 + c_1 r^2 + c_2 r^4 + \dots, \quad (1)$$

where μ_0 is a constant linear tune, c_1, c_2 are the coefficients of the higher order terms in the expansion of the dependence of the tune μ on the particle's amplitude in the normal form coordinates, where the amplitude is defined to be $r = \sqrt{(t^+)^2 + (t^-)^2}$ for the particle with normal form coordinates (t^+, t^-).

The value of μ_0 is also known for each pair of the conjugate variables describing the transversal dynamics, while there is not enough information to find the coefficients in the expansion (1). The task is to find the connection between the number of turns N required for the amplitude of the central particle after the kick $r(N)$ to fall to half of its value before the kick.

OBJECTIVE FUNCTION FOR THE STUDY

The purpose of the study is to restore at least the c_1 coefficient in the expansion (1) using the measurement results.

The objective function for the study is taken to be of the form

$$J(c_1, c_2) = |\eta(c_1, c_2) - N|, \quad (2)$$

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where $\{\eta : r|_\eta \leq \frac{1}{2}r|_0\}$, $r|_\eta$ is the beam center of mass amplitude after η turns, and $r|_0$ is the beam center of mass amplitude immediately after the kick (the peak amplitude).

η is the function of r, and in turn, r is the function of $x_c^{(N)}, y_c^{(N)}$, where $x_c^{(N)}$ and $y_c^{(N)}$ are the coordinates of the center of mass after N turns after the kick in the normal form coordinates.

CALCULATION RESULTS VERSUS MEASUREMENT RESULTS

The normal form transformation is a nonlinear change of coordinates, such that after the transformation the dynamics of the particles is represented in a very systematic way. The details of the transformation algorithm can be found in [3]. The most important part for this study is that after the normal form transformation all the particles follow circles with angular velocity depending on the amplitude. This is the key fact allowing to establish a connection between the nonlinear tune shift with amplitude and the behavior of the beam.

As a rule, $c_1 r^2$ is the dominating term in the expansion (1). Hence, finding the coefficient c_1 is the most important part of the problem.

In the nonlinear case the function connecting the initial and final coordinates of the particles after one full revolution (called the transfer map) has the form:

$$\mathcal{M} = \begin{pmatrix} \cos 2\pi\mu(r) & -\sin 2\pi\mu(r) \\ \sin 2\pi\mu(r) & \cos 2\pi\mu(r) \end{pmatrix}. \tag{3}$$

If the transfer map $\mathcal{M}$ is known, one can track the behavior of particles for arbitrary many turns. That, in turn, allows to find the number of turns corresponding to the moment when the center of mass amplitude at the half of its value right after the kick, N. This establishes the connection between c_1 and N (in the form of the objective function (2)). The number N can be found from the measurements (Fig. 1).

Hence, the problem under consideration has been reduced to establishing a dependence of N on various values of c_1 and R.

ELLIPTICAL BEAM, NORMAL OR ARBITRARY DISTRIBUTION

Let us assume that the beam has an elliptical shape. Then after the transformation to the normal form coordinates this beam has the elliptical shape again, and the axes of the transversal section of the beam are equal, hence the boundary curve for the beam in the normal form coordinate pair is a circle, and the parametric representation for it can be found in the form of the equations for two half-circles: $(r, \varphi_1(r))$, $(r, \varphi_2(r))$. Without loss of generality it can be assumed that the resulting circle has its center on the horizontal axis, with the coordinates $(d, 0)$, where $d > 0$ is known (the angle can be changed as only the radius is the quantity of interest). Let ρ be the radius of the beam, then

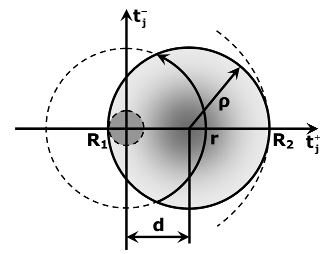

Figure 2: $R_1 = d - \rho < 0$.

the beam lies between $R_1 = d - \rho$ and $R_2 = d + \rho$ Both d and ρ parameters can be found by applying the linear normal form transformation to the displaced beam boundaries.

It often happens that the radius R_1 is less than zero, which means that the origin (0,0) gets inside the beam. In the special case of $R_1 = d - \rho < 0$ with the layout corresponding to Fig. 2, for $0 < r < |R_1|$ the whole contour $(r, \varphi \in [-\pi, \pi))$ belongs to the beam, and one can assume for such r that φ goes from $-\pi$ to π.

After N turns each particle of the distribution will have the phase advance of

$$\theta_N(r) = 2\pi N\mu(r) = 2\pi N(\mu_0 + c_1 r^2 + c_2 r^4)$$

(orders up to 4 are taken into account). Hence, the particle with radius $R_1 < r < R_2$ located on the front (back) line of the distribution will have a phase difference of $\Delta\theta_N(r) = 2\pi N(\mu(r) - \mu(R_1))$ with respect to the inner particle.

Assume that the beam distribution is normal in both directions in every pair of coordinates, and each two directions are independent. As the beam is round in the normal form coordinates, the variances in both eigen-directions are the same $\sigma = \sigma_x = \sigma_y$, and hence the resulting density of the bivariate distribution is defined by the formula

$$f(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-((x-d)^2 + y^2)}{2\sigma^2}\right),$$

as the mean values for the distribution are d and 0. Note that this formula is only valid for the initial distribution, when $\theta_N = 0$, and after N turns θ_N should be subtracted from the value of the angle.

To find the centroid of any planar figure, three integral formulas can be employed:

$$\begin{cases} S = \displaystyle\iint r dr d\theta; \\ x_c = \displaystyle\frac{1}{S}\iint r^2 \cos\theta dr d\theta; \\ y_c = \displaystyle\frac{1}{S}\iint r^2 \sin\theta dr d\theta. \end{cases}$$

x_c and y_c are the coordinates of the center of mass in the chosen coordinate system.

For the case under consideration after certain transformations that can be found in [4], the expressions for S,

$x_c^{(N)}$, and $y_c^{(N)}$ are

$$\left\{\begin{aligned} S &= \int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r f(\theta-\theta_N)d\theta dr; \\ x_c^{(N)} &= \frac{1}{S}\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2\cos\theta f(\theta-\theta_N)d\theta dr; \\ y_c^{(N)} &= \frac{1}{S}\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2\sin\theta f(\theta-\theta_N)d\theta dr. \end{aligned}\right. \tag{4}$$

NUMERICAL EXPERIMENT RESULTS

The calculation method described above allows one to find the dependence $r = r(N, c_1, c_2)$ for the elliptical beams with arbitrary particle distributions, the only requirement being that the initial distribution density function is known. Hence, for each pair of values c_1 and c_2, one can find $\eta(c_1, c_2)$ introduced in the previous section as well as the corresponding values of the function (2). Having these data available and employing various optimization methods, one can find the correct values of c_1 based on one particular measurement or both coefficients c_1 and c_2, provided that measurements for different kick strengths are available.

The numerical results for the Tevatron correspond to the values obtained by tracking the nonlinear model of the machine. The number of turns after which the amplitude of the center of mass falls down to a half of its value varies depending on the BPM, one of the total of 115 reliable measurements. Taking the average over all the BPMs one obtains that $N \approx 1000$.

The optimization procedure returns the expected value of $c_1 = -2511$ for the initial beam amplitude after the kick of $r = 0.24 \cdot 10^{-3}$. Taking into account that $\mu_0 = 0.585$, one gets

$$\mu \approx \mu_0 + c_1 r^2 = 0.585 - 1.4463 \cdot 10^{-4}. \tag{5}$$

To conceive how close the obtained value of c_1 is to the realistic value of the tune shift with amplitude, a comparison was performed in COSY INFINITY [5] using the nonlinear model of the Tevatron available at the official lattice page at Fermilab [1]. The COSY calculation shows that the expected value of c_1 for the nonlinear model should be -2541, which means the calculated value found by the optimization differs from the model value by not more than 2%. At the same time, only the information about the distribution of the particles in the beam, the size of the beam, and the linear dynamics was used to find the nonlinear tune shift. Necessary additional information was extracted from the measurements.

Figure 3 shows the graphs of the calculated amplitude with $c_1 = -2511$ and the model amplitude with $c_1 = -2541$. The slight difference between the graphs can be explained not only by using different c_1's, but also by the fact that the fourth order term $c_2 r^4$ in the expansion of μ has not been taken into account. At the same time, the

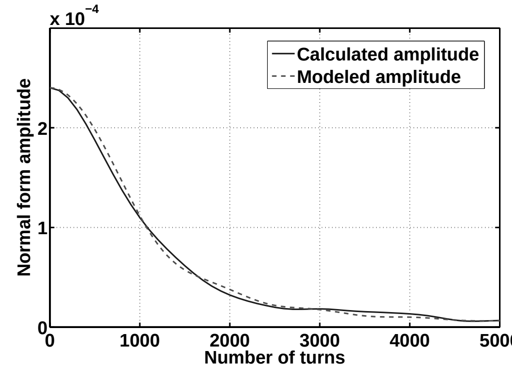

Figure 3: Calculation results and the comparison with the nonlinear model.

similarity of the graphs allows to conclude that the model represents the real machine quite accurately, at least for the low order nonlinearities.

Also, the validity of the approach studied is perfectly supported by the independent calculations done years ago. There is an estimate of the nonlinear tune shift by R.Meller et al. [6], given by the following formula:

$$\mu \approx \mu_0 - \kappa A^2, \quad \kappa \approx \frac{1}{4\pi N},$$

where A is the amplitude of the center of mass of the beam, measured in σ units of the beam under consideration. This formula is derived for the beams with a normal distribution of the particles, and it represents a good approximation when the transversal kick is relatively weak.

The value comparison of κA^2 from Meller's article to the value of $c_1 r^2$, obtained by the calculation using the algorithm described above, gives the following results: $\kappa = 7.96 \cdot 10^{-5}$, $A = 1.36$,

$$\mu \approx \mu_0 + \kappa A^2 = 0.585 - 1.4723 \cdot 10^{-4}, \tag{6}$$

that is, the difference between the values obtained using different approximations in Eqs. (5) and (6) is less than 2%.

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