Extended Soft-Gaussian Code for Beam-Beam Simulations

D. Xu[1] Y. Luo [1] C. Montag[1] Y. Hao[2] J. Qiang[3]

[1]Brookhaven National Lab [2]Michigan State University [3]Lawrence Berkeley National Laboratory

Introduction

- Strong-strong simulation is used to study the coherent motion during beam-beam collision
- PIC based strong-strong simulation is self-consistent, but much noisy
- Soft-Gaussian model assumes both beams being perfect bi-Gaussian distribution. Although it is faster and less noisy, the assumption may be oversimplified
- The discrepancy between the weak-strong and strong-strong simulation for Electron-Ion Collider (EIC) has been found. It is important to understand the difference in case there is some coherent mechanism shadowed by the large numerical noise.
- In the extended soft-Gaussian model (ESG), not only $\sigma_{x,y}$, but also σ_{xy} and 3rd order moments are considered. The ESG would be a better benchmark tool for strong-strong simulation.

Standard soft-Gaussian model

The beam-beam potential generated by an upright bi-Gaussian distribution is,

$$U_g = \frac{Q_1 Q_2 N r_0}{\gamma_0}\int_0^\infty \mathrm{d}u \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+u}-\frac{y^2}{2\sigma_y^2+u}\right)}{\sqrt{2\sigma_x^2+u}\sqrt{2\sigma_y^2+u}} \tag{1}$$

where N is the total particle number, $r_0 = e^2/(4\pi\epsilon_0 mc^2)$ the classical radius, γ_0 the relativistic factor of the test particle, $Q_{1,2}$ the charge numbers of particles from two colliding bunches, and $\sigma_{x,y}$ are the RMS beam sizes at the collision point.
The deflection angle from the above bi-Gaussian beam can be obtained from the well-known Bassetti-Erskine formula,

$$U_y + \mathrm{i}U_x = -\frac{Q_1Q_2Nr_0}{\gamma_0}\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\left[w\left(\frac{x+\mathrm{i}y}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right) - w\left(\frac{\frac{\sigma_y}{\sigma_x}x+\mathrm{i}\frac{\sigma_x}{\sigma_y}y}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\right] \tag{2}$$

where $U_{x,y}$ is the abbreviation of the derivative

$$U_x = \frac{\partial U_g}{\partial y}, \qquad U_y = \frac{\partial U_g}{\partial y} \tag{3}$$

x, y the coordinates of the test particle, and $w(z)$ is the Faddeeva function,

$$w(z) \equiv \exp\left(-z^2\right)\left(1+\frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^z \mathrm{d}t\ \mathrm{e}^{t^2}\right) \tag{4}$$

Including beam tilt

A general 2D Gaussian distribution can be described by its Σ matrix,

$$\Sigma = \begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix} \tag{5}$$

$$\phi_g(x,y) = \frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left[-\frac{1}{2}(x,y)\,\Sigma^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right] \tag{6}$$

To use the Bassetti-Erskine formula, we can apply a rotation on the coordinates (x, y)

$$A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}, \qquad \begin{bmatrix}\overline{x}\\ \overline{y}\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix} \tag{7}$$

so that the Σ matrix in the rotated frame is diagonal,

$$\begin{bmatrix}\sigma_{yy} & -\sigma_{xy}\\ -\sigma_{xy} & \sigma_{xx}\end{bmatrix} = A^{\mathrm{T}}\begin{bmatrix}\overline{\sigma}_{yy} & 0\\ 0 & \overline{\sigma}_{xx}\end{bmatrix}A \tag{8}$$

where the overline denotes the variable in the rotated frame.
A possible solution is

$$\cos 2\theta = \frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\sigma_{xy}^2}} \tag{9}$$

$$\sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\sigma_{xy}^2}} \tag{10}$$

$$\overline{\sigma}_{xx} = \frac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right)+\frac{1}{2}\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\sigma_{xy}^2} \tag{11}$$

$$\overline{\sigma}_{yy} = \frac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right)-\frac{1}{2}\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\sigma_{xy}^2} \tag{12}$$

Rotating back to the original $x-y$ frame, the deflection angle by beam-beam interaction is

$$\begin{bmatrix}U_x\\U_y\end{bmatrix} = A^{-1}\begin{bmatrix}\overline{U}_x\\ \overline{U}_y\end{bmatrix} \tag{13}$$

where $\overline{U}_{x,y}$ is calculated from Eq. (2) with substitution of $\overline{x}, \overline{y}, \overline{\sigma}_{x,y}$.

Hermite Polynomial

The Hermite polynomial is defined as

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(-\frac{x^2}{2}\right) \tag{14}$$

The orthogonality,

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\exp\left(-\frac{x^2}{2}\right)\,\mathrm{d}x = \sqrt{2\pi}n!\delta_{mn} \tag{15}$$

The first seven Hermite polynomials are:

$$H_0(x) = 1,\quad H_1(x) = x,\quad H_2(x) = x^2-1,\quad H_3(x) = x^3-3x$$
$$H_4(x) = x^4-6x^2+3,\quad H_5(x) = x^5-10x^3+15x,\quad H_6(x) = x^6-15x^4+45x^2-15$$

Including 3rd order moments

After the frame is tilted, we can further extend the model to include higher order moments with the help of Hermite polynomial,

$$\phi(x,y) = a_{ij}H_i(x/\sigma_x)H_j(y/\sigma_y)\,\phi_g(x,y) \tag{16}$$

where the repeated indices mean summation, $\phi_g(x,y)$ the standard bi-Gaussian kernel as shown in Eq. (6).
The coefficient is

$$\begin{aligned} a_{mn} &= \frac{1}{m!n!}\int H_m(x/\sigma_x)H_n(y/\sigma_y)\,\phi(x,y)\ \mathrm{d}x\mathrm{d}y\\ &= \frac{1}{m!n!}\left\langle H_m(x/\sigma_x)H_n(y/\sigma_y)\right\rangle \end{aligned} \tag{17}$$

where the angle bracket means taking the average over all macro particles in simulation.
In the rotated frame, the first two orders are corrected zero,

$$\begin{aligned} & a_{00} = 1,\quad a_{10} = a_{01} = a_{20} = a_{11} = a_{02} = 0\\ & a_{30} = \frac{1}{6}\left\langle\frac{x^3}{\sigma_x^3}\right\rangle,\qquad a_{21} = \frac{1}{2}\left\langle\frac{x^2y}{\sigma_x^2\sigma_y}\right\rangle\\ & a_{03} = \frac{1}{2}\left\langle\frac{xy^2}{\sigma_x\sigma_y^2}\right\rangle,\qquad a_{03} = \frac{1}{6}\left\langle\frac{y^3}{\sigma_y^3}\right\rangle \end{aligned} \tag{18}$$

Upto 3rd order, the beam-beam potential is

$$U = U_g - a_{30}\sigma_x^3U_{xxx} - a_{21}\sigma_x^2\sigma_yU_{xxy} - a_{12}\sigma_x\sigma_y^2U_{xyy} - a_{03}\sigma_y^3U_{yyy} \tag{19}$$

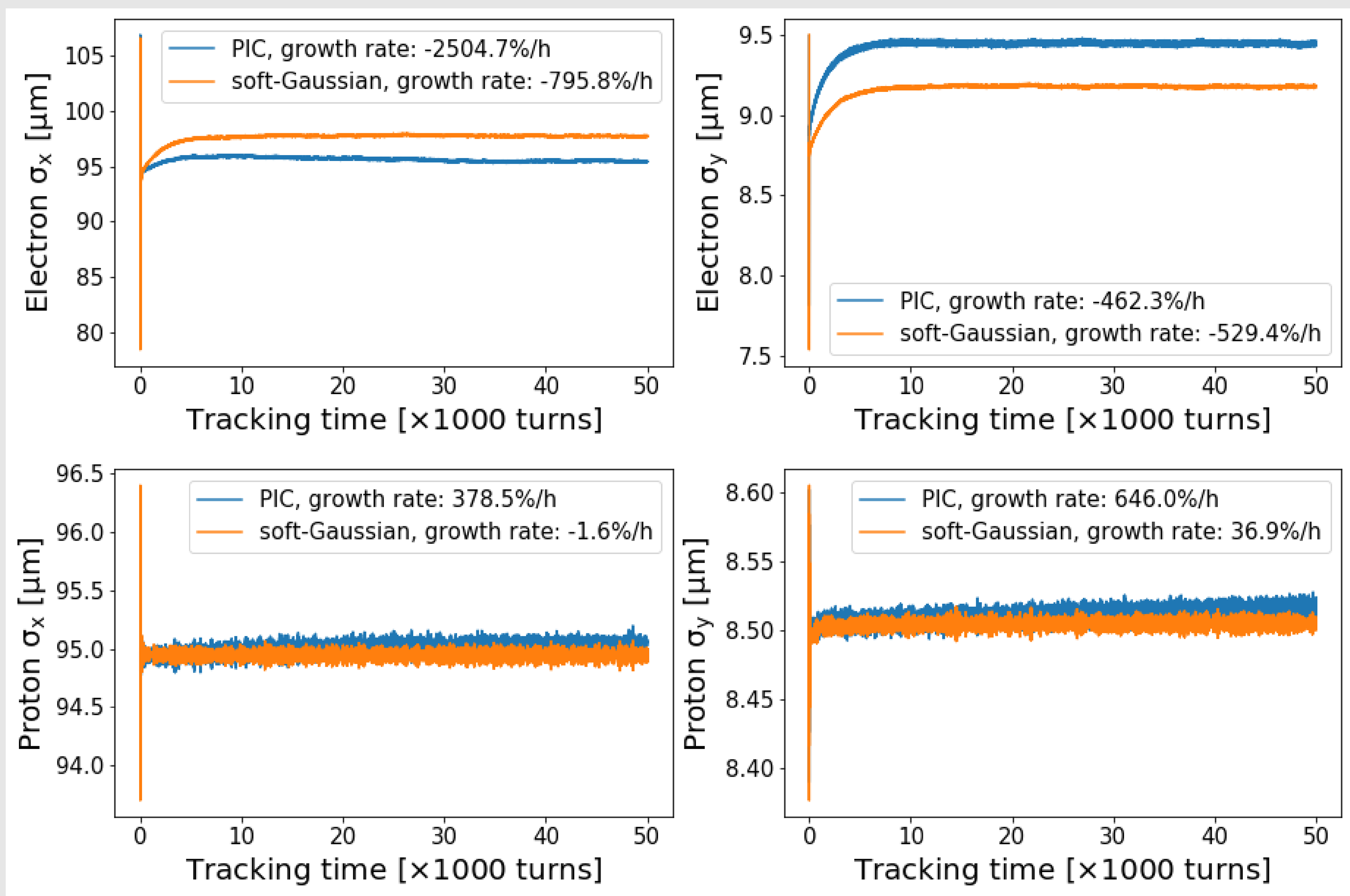
where $U_{xxx}, U_{xxy}, U_{xyy}, U_{yyy}$ on the right hand are the partial derivatives of U_g which can be obtained analytically.
In our code, there are 20 terms of 3rd order moments calculated at the IP. Assuming the drift length between the collision point and the IP is L, the moments at collision point are

$$\begin{aligned} < x^3 > =& < x_0^3 > +3 < x_0^2p_{x0} > L + 3 < x_0p_{x0}^2 > L^2\\ &+ < p_{x0}^3 > L^3\\ < x^2y > =& < x_0^2y_0 > + < x_0^2p_{y0} > L + 2 < x_0p_{x0}y_0 > L\\ &+ 2 < x_0p_{x0}p_{y0} > L^2 + < p_{x0}^2y_0 > L^2\\ &+ < p_{x0}^2p_{y0} > L^3\\ < xy^2 > =& < x_0y_0^2 > +2 < x_0y_0p_{y0} > L + < x_0p_{y0}^2 > L^2\\ &+ < p_{x0}y_0^2 > +2 < p_{x0}y_0p_{y0} > L^2\\ &+ < p_{x0}p_{y0}^2 > L^3\\ < y^3 > =& < y_0^3 > +3 < y_0^2p_{y0} > L + 3 < y_0p_{y0}^2 > L^2\\ &+ < p_{y0}^3 > L^3 \end{aligned} \tag{20}$$

the subscript "0" means the average is calculated at the IP.

Simulation: soft-Gaussian vs PIC

The growth rate is linearly fitted from the last 60% tracking data



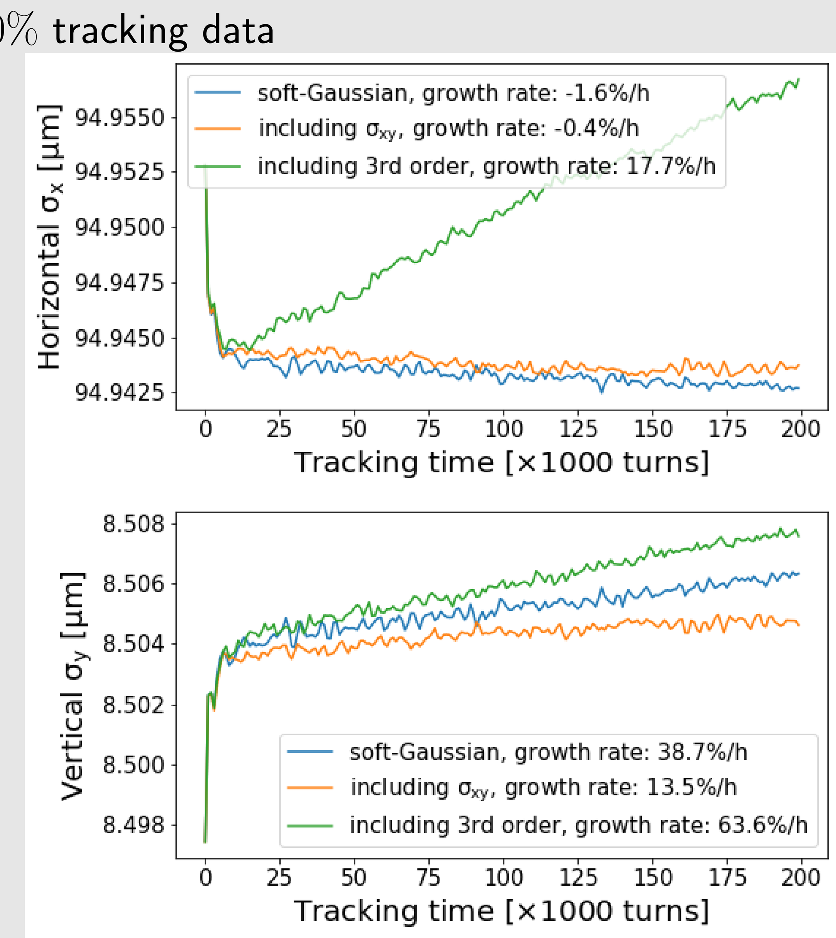
The equilibrium electron sizes are different in both codes because the soft-Gaussian is not self-consistent. Compared with BeamBeam3D, the soft- Gaussian code is less noisy.

Simulation: extended soft-Gaussian

The growth rate is linearly fitted from the last 60% tracking data

Table 1: Flat Beam Parameters in the EIC CDR

Quantity	unit	proton	electron
Crossing angle	mrad	25	
Beam energy	GeV	275	10
Bunch intensity	10^{11}	0.668	1.72
β^* at IP	cm	80/7.2	55/5.6
Beam sizes at IP	μm	95/8.5	
Bunch length	cm	6	2
Transverse tunes		0.228/0.210	0.08/0.06
Longitudinal tune		0.01	0.069



Although the growth number is small, the 3rd order moments contribute to the horizontal and vertical size growth.

 dxu@bnl.gov