Error Minimization in Transverse Phase-Space Measurements Using Quadrupole and Solenoid Scans

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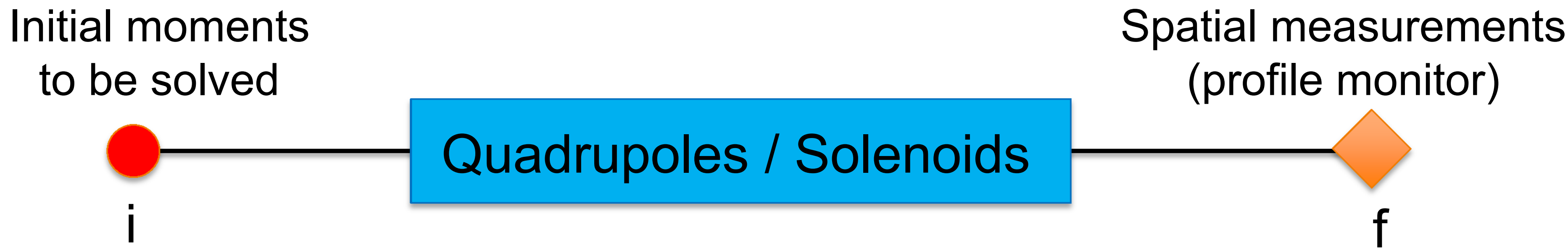
Quadrupole and Solenoid Scans



Solve linear system $A\boldsymbol{x} = \boldsymbol{b}$

- A: coefficient matrix derived from linear transfer maps
  - Misalignment errors, field errors etc.
- $\boldsymbol{b}$: measurement results
  - Measurement errors
- $\boldsymbol{x}$: solution – how to quantify and minimize its errors?

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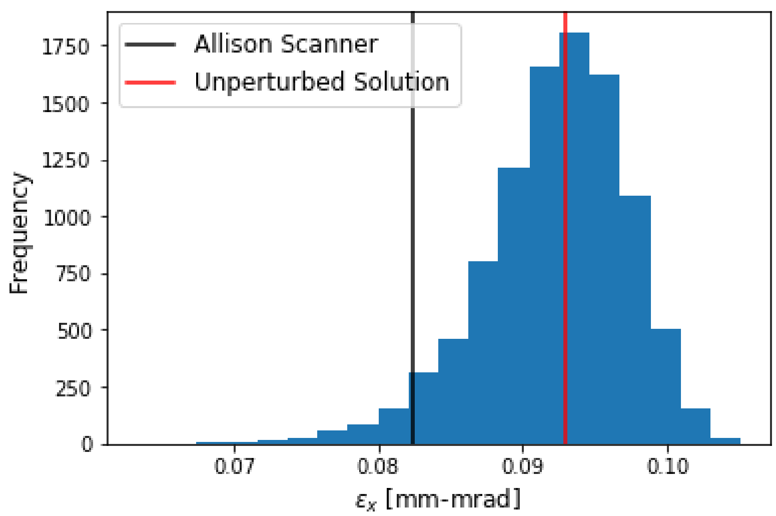
Error Quantification: 1) Statistical Analysis

- Solve $(A + \delta A)(\boldsymbol{x} + \delta \boldsymbol{x}) = (\boldsymbol{b} + \delta \boldsymbol{b})$ repeatedly
- Apply different perturbations δA and $\delta \boldsymbol{b}$ each time
  - Perturbation magnitudes based on estimates on error sources
- Frequency count of solutions $(\boldsymbol{x} + \delta \boldsymbol{x})$ gives probability distribution



- Can obtain actual error values

Error Quantification: 2) Sensitivity Analysis

- How sensitive is the solution to perturbations?
- Key parameter:

$$\text{condition number } \kappa(A) = \|A\| \, \|A^{+}\| = \frac{\sigma_1(A)}{\sigma_n(A)} \geq 1$$

$\sigma_1(A)$: largest singular value of A $\qquad$ $\sigma_n(A)$: smallest singular value of A

- Expression that bounds the maximum relative change in the solution:

$$\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)^2 \frac{\|\mathbf{r}\|}{\|A\mathbf{x}\|} \frac{\|\delta A\|}{\|A\|} + \kappa(A) \left(\frac{\|\mathbf{b}\|}{\|A\mathbf{x}\|} \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|} + \frac{\|\delta A\|}{\|A\|} \right)$$

Relative change in solution

Relative error in transfer matrices

Relative error in measurements

$\rightarrow 0$ as $A\mathbf{x} - \mathbf{b} \rightarrow 0$

$\rightarrow 1$ as $A\mathbf{x} - \mathbf{b} \rightarrow 0$

- Can determine which system's solutions have sharper probability distributions without explicitly calculating error values

How to Minimize Condition Number in Quad Scan

- GSI: emphasized importance of low condition number, but did not specify how it can be achieved

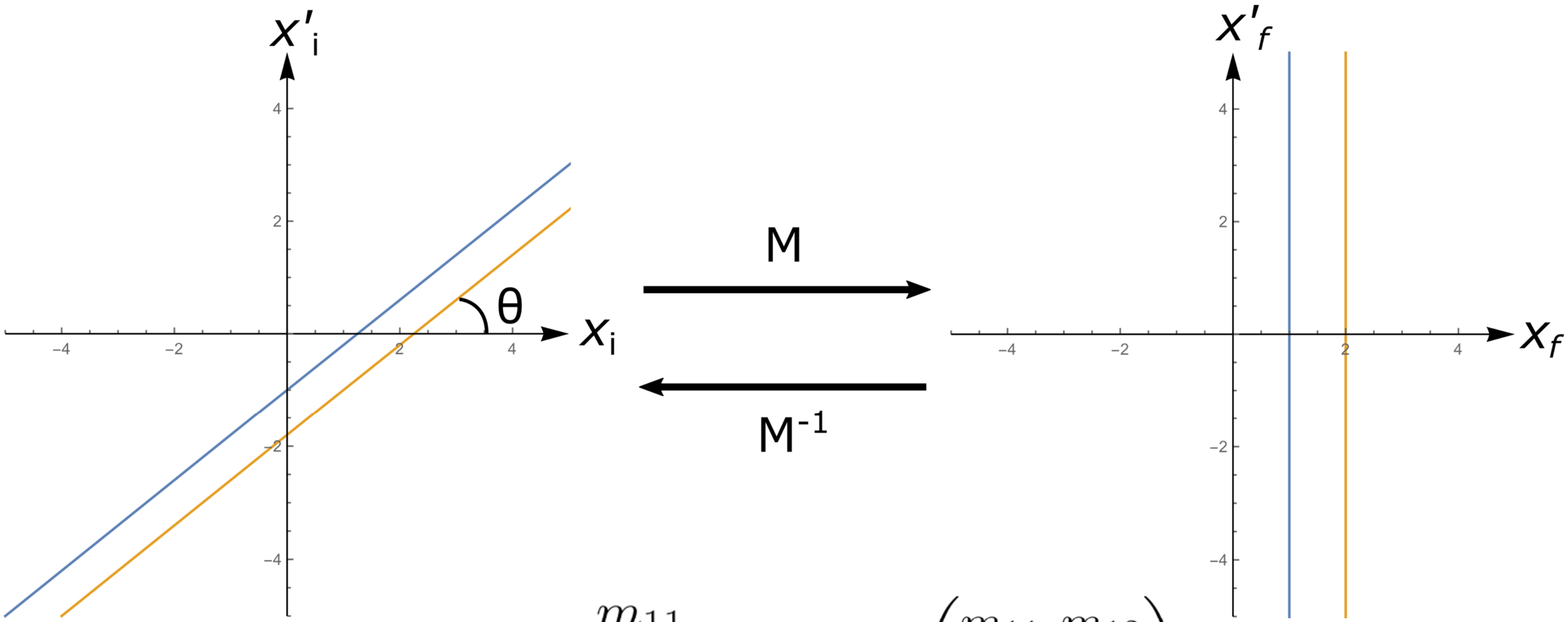
- Enumerative search impractical
  - Suppose each quad in a doublet can attain 10 values: 100 doublet settings
  - Possible combinations for a 4-measurement quad scan:

$$\binom{100}{4} = \frac{100!}{4! \times 96!} \approx 4 \times 10^6$$

  - Too computationally intensive

- PSI: choose quad parameters corresponding to discrete steps in phase advance
  - Phase advance correspond to rotation angle in normal coordinates
  - Rotation in phase space depends on orientation of invariant ellipse
  - Invariant ellipse may not exist

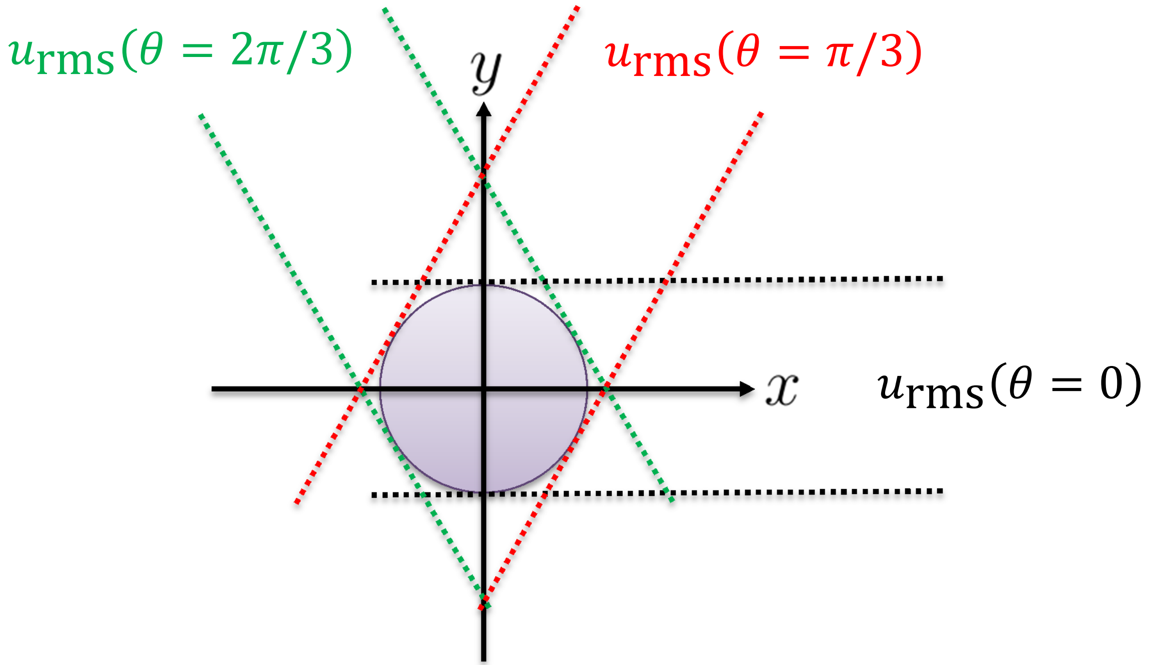
Direct Picture: Projection Angle Corresponding to Transfer Map



$$\tan\theta = -\frac{m_{11}}{m_{12}} \qquad \mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

- Each linear map gives a projection of the initial phase space
  - Choose quad parameters corresponding to diverse projection angles
  - E.g. with 100 doublet settings, calculate all θ and pick diverse set of 4
  - Much more efficient than computing all possible combinations

Intuitively: Measure from Many Different Angles



$$\begin{pmatrix} \cos^2(\theta_1) & 2\sin(\theta_1)\cos(\theta_1) & \sin^2(\theta_1) \\ \cos^2(\theta_2) & 2\sin(\theta_2)\cos(\theta_2) & \sin^2(\theta_2) \\ \vdots & \vdots & \vdots \\ \cos^2(\theta_n) & 2\sin(\theta_n)\cos(\theta_n) & \sin^2(\theta_n) \end{pmatrix} \begin{pmatrix} \langle xx \rangle \\ \langle xy \rangle \\ \langle yy \rangle \end{pmatrix} = \begin{pmatrix} \langle u(\theta_1)\, u(\theta_1) \rangle \\ \langle u(\theta_2)\, u(\theta_2) \rangle \\ \vdots \\ \langle u(\theta_n)\, u(\theta_n) \rangle \end{pmatrix}$$

$$\theta_j = \phi + \frac{j\pi}{n} \qquad u_j = x\cos(\theta_j) + y\sin(\theta_j)$$

Evenly Distributed Angles are Almost Perfect

- For all $n \geq 3$, $\kappa(A) = \sqrt{2} \approx 1$

- One case that is certainly as insensitive to perturbations as possible
- Establishes a target in the search for optimal scan parameters

$$\overbrace{\begin{pmatrix} \cos^2(\theta_1) & 2\sin(\theta_1)\cos(\theta_1) & \sin^2(\theta_1) \\ \cos^2(\theta_2) & 2\sin(\theta_2)\cos(\theta_2) & \sin^2(\theta_2) \\ \vdots & \vdots & \vdots \\ \cos^2(\theta_n) & 2\sin(\theta_n)\cos(\theta_n) & \sin^2(\theta_n) \end{pmatrix}}^{A} \begin{pmatrix} \langle xx \rangle \\ \langle xy \rangle \\ \langle yy \rangle \end{pmatrix} = \begin{pmatrix} \langle u(\theta_1)\, u(\theta_1) \rangle \\ \langle u(\theta_2)\, u(\theta_2) \rangle \\ \vdots \\ \langle u(\theta_n)\, u(\theta_n) \rangle \end{pmatrix}$$

$$\theta_j = \phi + \frac{j\pi}{n} \qquad u_j = x\cos(\theta_j) + y\sin(\theta_j)$$

How to Attain Multi-Angle Projection in Practice

- In practice, a quadrupole scan gives:

$$\begin{pmatrix} a^{(1)}\cos^2\left[\theta^{(1)}\right] & 2a^{(1)}\sin\left[\theta^{(1)}\right]\cos\left[\theta^{(1)}\right] & a^{(1)}\sin^2\left[\theta^{(1)}\right] \\ a^{(2)}\cos^2\left[\theta^{(2)}\right] & 2a^{(2)}\sin\left[\theta^{(2)}\right]\cos\left[\theta^{(2)}\right] & a^{(2)}\sin^2\left[\theta^{(2)}\right] \\ \vdots & \vdots & \vdots \\ a^{(n)}\cos^2\left[\theta^{(n)}\right] & 2a^{(n)}\sin\left[\theta^{(n)}\right]\cos\left[\theta^{(n)}\right] & a^{(n)}\sin^2\left[\theta^{(n)}\right] \end{pmatrix} \begin{pmatrix} \langle xx\rangle \\ \langle xx'\rangle \\ \langle x'x'\rangle \end{pmatrix} = \begin{pmatrix} \langle xx\rangle^{(1)} \\ \langle xx\rangle^{(2)} \\ \vdots \\ \langle xx\rangle^{(n)} \end{pmatrix}$$

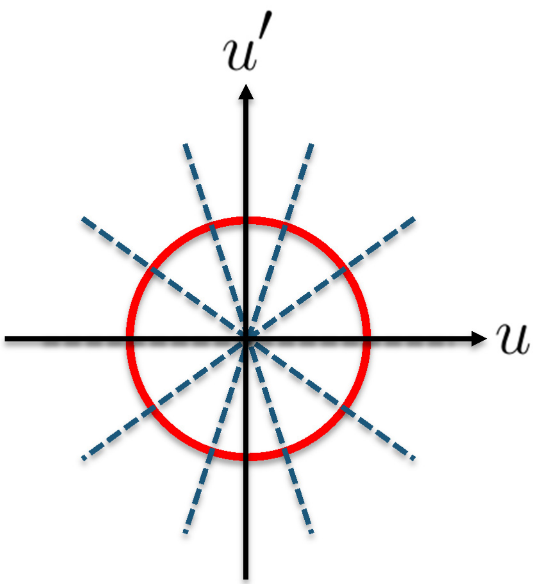
$$a^{(j)} = \sqrt{m_{11}^{2(j)} + m_{12}^{2(j)}} \qquad \theta^{(j)} = \arctan\left[-m_{11}^{(j)}/m_{12}^{(j)}\right] = \phi + \frac{j\pi}{n} + \psi_j$$

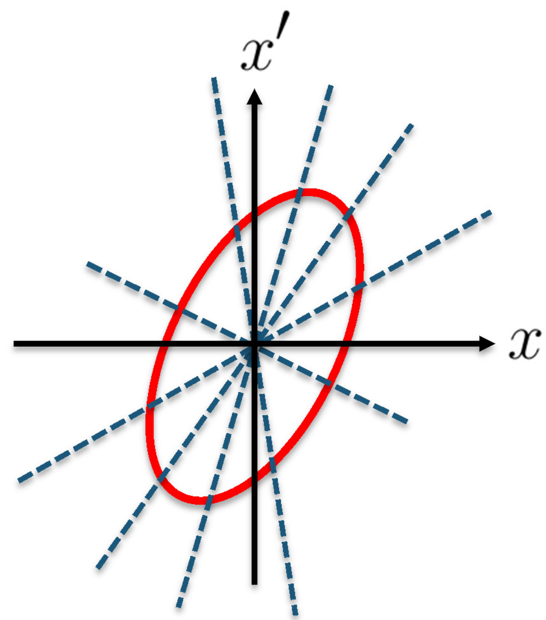
- Each $a^{(j)}$ factor is different
- $\psi^{(j)}$ are deviations from ideal angle distribution

- To maximally resemble ideal system: $\text{minimize } \left(\frac{\max\left[a_j\right]}{\min\left[a_j\right]} - 1\right) \text{ and } \psi_j\text{'s}$

Initial Phase Space in What Units?

- Projection angle $\theta = \tan^{-1}(-m_{11}/m_{12})$ depends on choice of units
- No reference is made to the initial beam distribution if one simply requests $\theta^{(j)} = \phi + j\pi/n$
- Natural choice of coordinates: one in which beam is round
  - Put errors in both dimensions on an equal footing
  - Projection angles are defined by M^*
  - If A close to ideal, each measurement has roughly same size determined by the emittance





$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = R \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R^{\mathsf{T}}$$

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} R^{\mathsf{T}} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$M^* \begin{pmatrix} u \\ u' \end{pmatrix}_i = \begin{pmatrix} x \\ x' \end{pmatrix}_f$$

$$M^* = MR \begin{pmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{pmatrix}$$

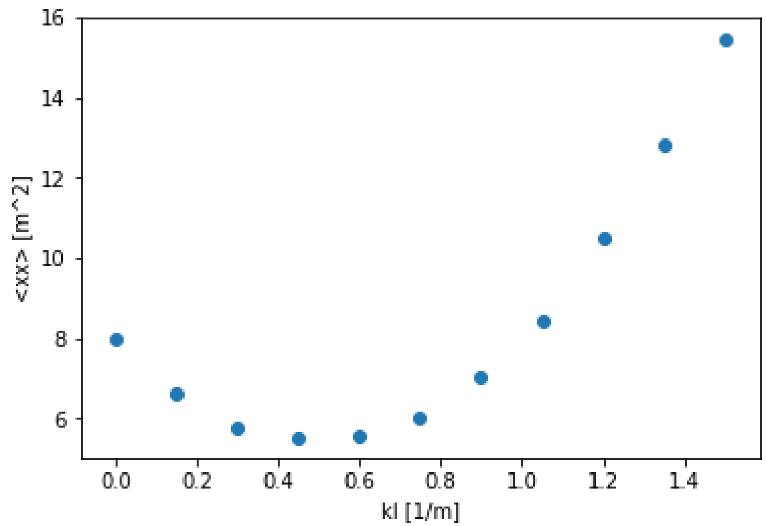
Conventional Single Quadrupole Scan with Parabolic Fit

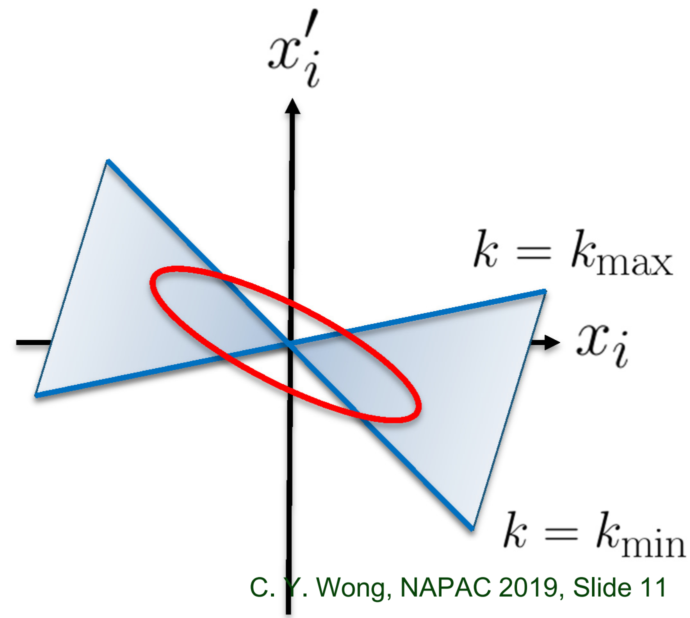
- Beam size vs. quadrupole strength
  - Parabolic (assuming thin lens)

- Why insist on having a trough?
  - Only exists if project both sides of ellipse
  - Indicates wide range of projection angles

- Potential problems
  - Large measurement errors at trough
  - Strong space charge effects at focus
  - Matrix A not optimally conditioned

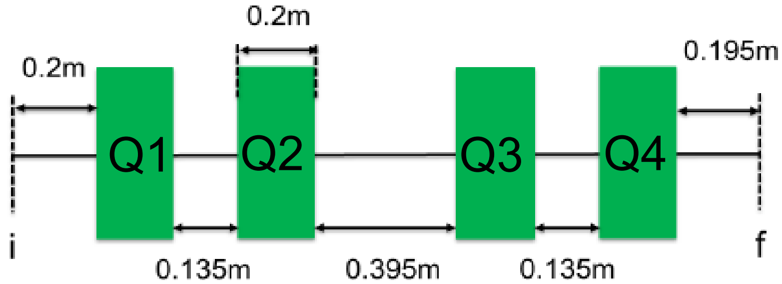
- Conversely, range of projection angles suggests favorable initial moments
  - Maybe very limiting if monitor close to quad (e.g. accelerator front ends)





Example: Two Sets of Quad Scan with Different Projection Angles

FRIB front end electrostatic quads



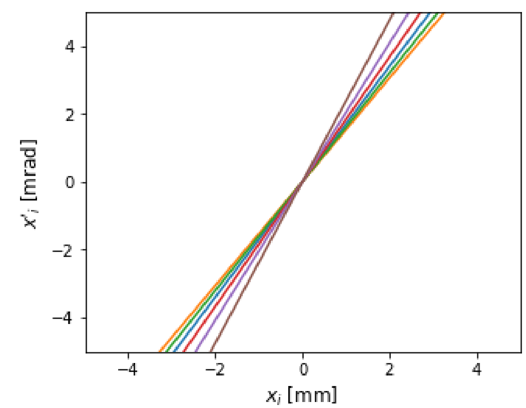
Wire scanner profile monitor

Measurement Set 1

Only the last quad is varied, routine choice

V1	-2657	-2657	-2657	-2657	-2657	-2657
V2	4513	4513	4513	4513	4513	4513
V3	-4295	-4295	-4295	-4295	-4295	-4295
V4	300	1300	2300	3300	4300	5300

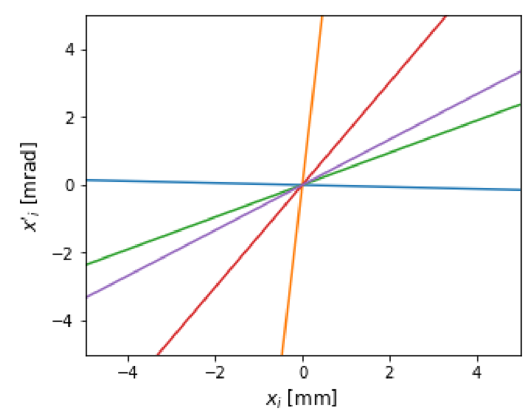
Projection Angles on Initial Phase Space



Measurement Set 2

Emphasize range of projection angles

V1	-500	-1500	-4000	-4000	-4500
V2	3000	4000	4500	5500	4500
V3	-4500	-4500	-4500	-3000	-4500
V4	4000	4000	4000	500	3500



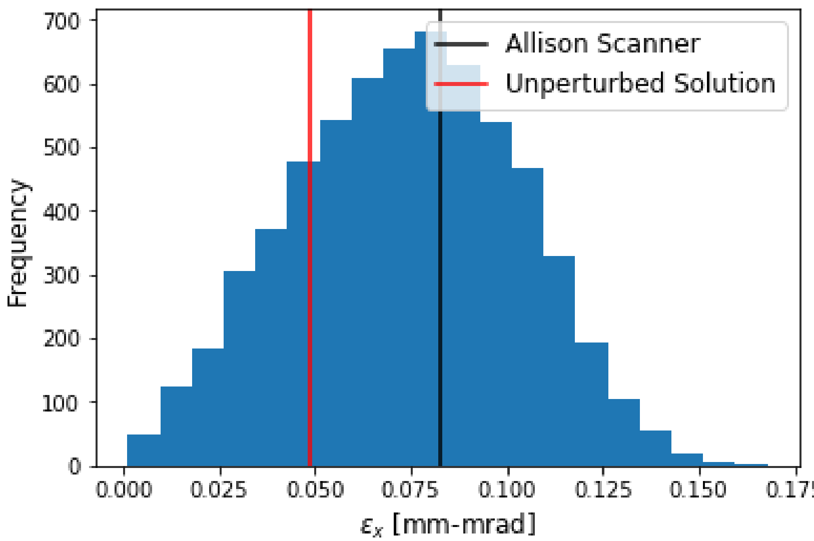
Error Quantification by Applying Random Measurement Errors

Measurement Set 1

Condition number $\kappa(A) = 466$

Apply random measurement error: $3\sigma = 1\%$

Std. Deviation of ε_x= 0.029 mm-mrad

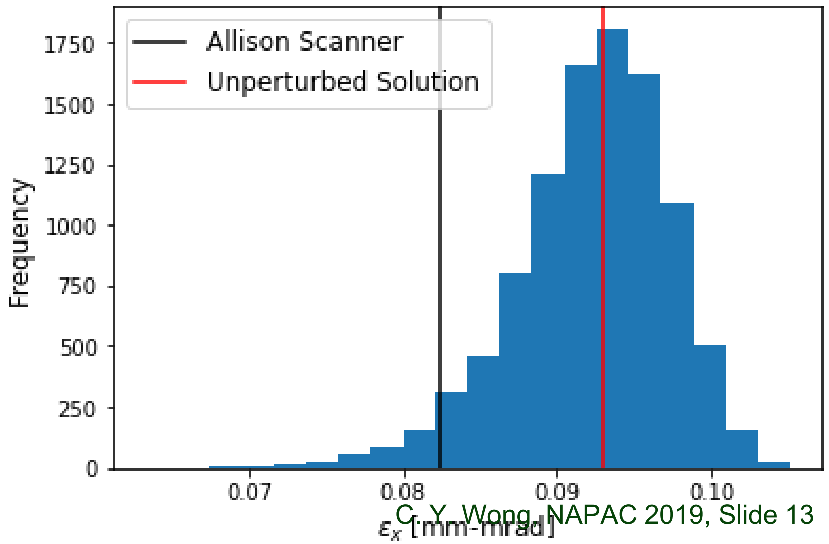


Measurement Set 2

Condition number $\kappa(A) = 14.5$

Apply random measurement error: $3\sigma = 10\%$

Std. Deviation of ε_x= 0.005 mm-mrad



Conclusion

- Two complimentary error quantifications: statistical and sensitivity
- Projection angle analysis sets clear target for error minimization
- Next: implement automated scan parameter selection at FRIB
- Analysis can be extended to 4D where trough-type arguments may not be readily available
  - xy-coupled moments from $\langle xy \rangle$ measurements and quadrupole scans
  - Solenoid scans

Thank you!