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METHOD FOR A MULTIPLE SQUARE WELL MODEL TO STUDY TRANSVERSE MODE COUPLING INSTABILITY*

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Abstract

In the high intensity limit it can become difficult to simulate intense beams sufficiently within a short time scale due to collective effects. Semi-Analytic methods such as the Square Well Model [1]/AirBag Square Well [2] (SWM/ABS) exist to estimate collective effects within a short time scale. SWM/ABS discretizes the longitudinal confining potential into a single square well enforcing linearity for the case of linear transverse optics. A method is proposed here to extend the Square Well Method to multiple square wells. This method preserves linearity properties that make it easily solvable within a short time scale as well as including nonlinear effects from the longitudinal potential shape.

INTRODUCTION

For the general case of instabilities in the presence of collective Space Charge (SC) and Wakefield effects, an exact solution does not exist except in special cases. To evaluate instabilities with SC and Wakefields, a simulation method must be applied. Unfortunately, many of these methods take a prohibitively long time to reach a solution. In order to reach a rapid solution while including both SC and Wakes, some sort of well founded simplification must be made to the dynamics of the system. Previous work exists where the longitudinal potential is simplified to a single finite square potential well [3]. A single potential well like this confers several useful properties to the longitudinal dynamics that make it easily solvable. This has been applied to the Transverse Mode Coupling Instability (TMCI).

For a single finite square well, there are two synchrotron tunes which correspond to a set of discrete longitudinal velocities in opposing longitudinal directions. These velocities form a single cycle with simple longitudinal particle dynamics. Under linear focusing optics collective particle moments and wakes are a system of linear ordinary differential equations solvable by matrix methods. These methods produce a set of tune shift parameters that indicate the presence of the TMCI instability. Although simple and efficient, the single finite square well underestimates the onset of TMCI compared to many other theoretical methods. [4]

By increasing the number of approximating square wells, the number of discrete longitudinal velocities increases allowing for the introduction of nonlinear chromatic effects as well as well as the introduction of synchrotron tune spread. This increases applicability to realistic systems.

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_x f \equiv \psi(t,z,\dot{z})$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_x x f \equiv D(t,z,\dot{z})$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_x p_x f \equiv P(t,z,\dot{z})$$

In the case of linear optics, these moments converge to a and we can apply ψ to the longitudinal Vlasov.

$$\frac{d\psi}{dt_L} = \frac{\partial\psi}{\partial t} + \dot{z}\frac{\partial\psi}{\partial z} - \frac{1}{m}\frac{dU(z)}{dz}\frac{\partial\psi}{\partial\dot{z}} = 0$$

The properties of these collective moments and their respective convective derivatives allow for the creation of collective equations of motion. These equations of motion are analogous to the single particle EOM in Eq. 1.

$$\frac{d^2D}{dt_L^2} = \frac{dP}{dt_L} = [g_1(z,\dot{z})D + g_2(t,z)\psi]\omega_0^2 \quad (2)$$

Single Sideband Approximation

In order to solve this problem within a fast timescale, we are looking turn these equations of motion into a series of 1st order differential equations that can be solved using matrix methods. There are two main methods one could take to accomplish this. The exact solution can be obtained by turning Eq. 2 into a set of two first order differential equations. This unfortunately increases the complexity of the solution and is more difficult to obtain physically meaningful solutions. The other main method that could be used is to restrict the problem to the upper betatron sideband [5]. This simplifies the moment D to a quickly oscillating term $e^{-iQ_x\omega_0 t}$ and a slowly varying term $\mathcal{D}$ where $D = Re(\mathcal{D}e^{-iQ_x\omega_0 t})$. The slowly varying term has a vanishing second derivative $\frac{d^2\mathcal{D}}{dt_L^2} \approx 0$. If this applied to equation 2 one will eventually obtain a linear differential equation as planned.

$$\frac{\partial\mathcal{D}}{\partial t} + \dot{z}\frac{\partial\mathcal{D}}{\partial z} = \frac{i\omega_0}{2Q_x}[\mathcal{D}(Q_x^2 + g_1(z,\dot{z})) + \psi e^{iQ_x t} g_2(t,z)] \quad (5)$$

In addition, collective terms such as $\mathcal{D}$ and ψ are of the form$\Sigma_{j=1}^{2m} f_j^n(t,z,\dot{z})\delta(\dot{z}-\dot{z}_j^n)$. Since these Differential equations exist at a set of discrete longitudinal velocities we can integrate over $\dot{z}$ and obtain a set of differential equations that can be used characterize the system.

Solving and Applying Continuity

With all the simplification out the the way Eqs. 7 and 8 represent a coupled set of Linear ODEs that span the longitudinal domain and include Space Charge effects along with Wakes. In addition, as these ODEs have constant coefficients, every domain slice can be solved by eigenvalue eigenvector methods of the form $\frac{\partial}{\partial z}\vec{V}^n = M\vec{V}^n$ with the vector made up of the basis functions of our differential equations.

$$\vec{V}^n = \begin{pmatrix} \hat{D}_1^n \\ \dots \\ \hat{D}_{2m}^n \\ \hat{F}_1^n \\ \dots \\ \hat{F}_\kappa^n \end{pmatrix}$$

Using this method we obtain a solution in terms of eigenvalues and eigenvectors of the following form.

$$\vec{V}^n(z) = \Sigma_{l=1}^{m+\kappa} \nu_l a_l e^{\Lambda_l(z-z_{n-1})}$$

where ν_l is the lth eigenvector, Λ_l the corresponding eigenvalue, and a_l are constraints chosen to obey boundary conditions. From here all that is left is to impose initial conditions. These are all fairly simple and corresponding to continuities in the spacial moment such as $D_j^n(z_n) = D_j^{n+1}(z_n)$ and continuity in the wake $F_k^n(z_n) = F_k^{n+1}(z_n)$. Other continuity equations exist, like the effect of wakes from previous bunches on the current bunch $F_k^1(0) = CF_k^{2N-1}(2N-1)$.

'Tune Search' for Constant Wake $w_0 = 1.0$, $w_0 = 2.5$

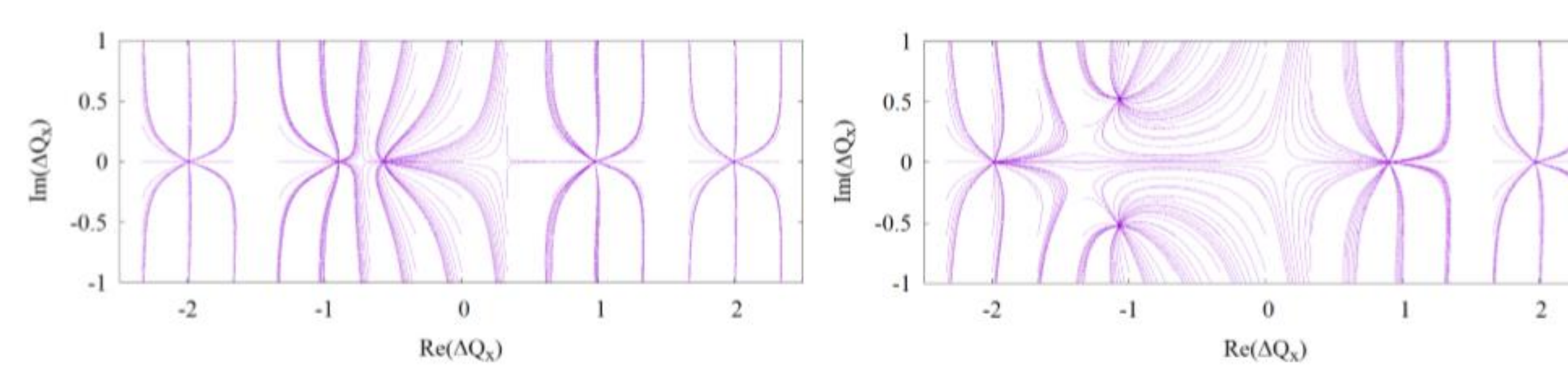

Figure 2: Example of tune shift algorithm searching for solutions for ΔQ_x that satisfy boundary conditions. The left plot only has real solutions and is below threshold for TMCI. The right plot has imaginary solutions which correspond to above threshold for TMCI.

Diagram of Multiple Square Well Model

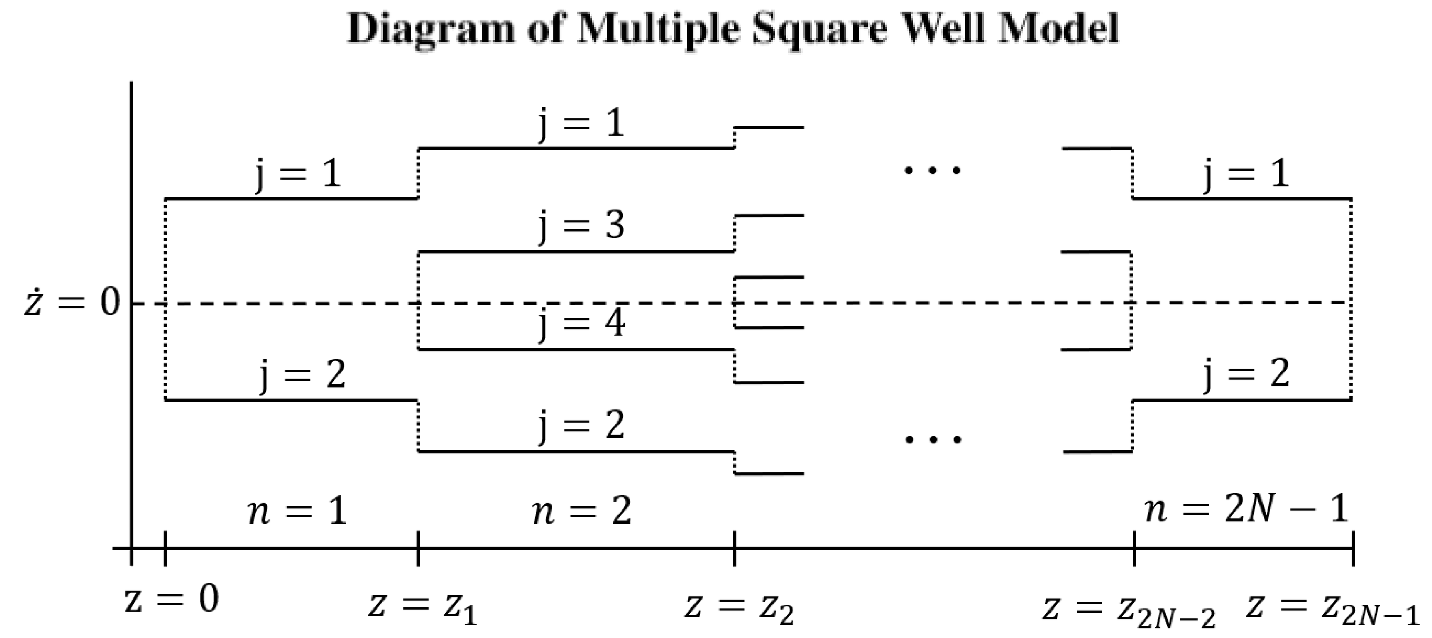

Figure 1: Longitudinal Phase Space diagram for a multiple square well method. Note the symmetry around $\dot{z}$ as well as the shape of the cycles and how the cycles are organized.

THEORY

To expand upon the SWM/ABS models one should begin with the equations of motion of a particle in a ring undergoing linear transverse motion with chromatic tune shift $\xi(\dot{z})$, linear space charge, and wakes (Eq. 2 from [3]).

$$\frac{d^2x}{dt^2} = \frac{dp_x}{dt} = [-(Q_x^2 + \xi(\dot{z}))x + C_{sc}\rho(z)(x - \overline{x}(t,z)) + \int_0^z dz' W(z-z')\rho(z')\overline{x}(t,z')]\omega_0^2 \equiv [xg_1(z,\dot{z}) + g_2(t,z)]\omega_0^2 \quad (1)$$

$\rho(z)$ corresponds to the linear longitudinal particle density. Due to the presence of both single particle motion and collective effects, it is difficult to treat analytically unless the collective effects act as a small perturbation on single particle motion.

To properly consider the collective effects, we shall define collective phase space moments from the Vlasov Equation. Since longitudinal dynamics generally vary slowly compared to transverse dynamics, we assume that the longitudinal dynamics are independent of transverse, while the transverse dynamics are dependent on longitudinal. This indicates that it is possible to construct and use two separate Vlasov Equations, one longitudinal and one transverse. The longitudinal particle density $\psi(t,z,\dot{z})$ can be constructed out of the moments from the transverse particle density $f(t,x,p_x,z,\dot{z})$

$$\frac{d^2D}{dt_L^2} = -(2iQ_x\omega_0\frac{d\mathcal{D}}{dt_L} + Q_x^2\omega_0^2\mathcal{D})e^{-iQ_x\omega_0 t} \quad (3)$$

Combining Eq. 2 and 3 we obtain the below statement.

$$\frac{d\mathcal{D}}{dt_L} = \frac{i\omega_0}{2Q_x}[\mathcal{D}(Q_x^2 + g_1(z,\dot{z})) + \psi e^{iQ_x t} g_2(t,z)]$$

Now we need to look at this problem in terms of partial derivatives to be more directly useful. Applying the longitudinal convective derivative:

$$\frac{\partial\mathcal{D}}{\partial t} + \dot{z}\frac{\partial\mathcal{D}}{\partial z} - \frac{1}{m}\frac{dU(z)}{dz}\frac{\partial\mathcal{D}}{\partial\dot{z}} = \frac{i\omega_0}{2Q_x}[\mathcal{D}(Q_x^2 + g_1(z,\dot{z})) + \psi e^{iQ_x t} g_2(t,z)] \quad (4)$$

Effects of Square Potential Well

In order to make the transverse collective equations of motion solvable as a system of linear differential equations, an arbitrary longitudinal potential must be simplified to a series of finite square potential wells. Since the longitudinal dynamics are independent of transverse directions (by assumption), we know that longitudinal particles will travel in filaments along with constant $\dot{z}$ within individual domain slices. This means we can further simplify Eq. 3 within a domain slice.

$$\int_{\dot{z}_j^n-\epsilon}^{\dot{z}_j^n+\epsilon} d\dot{z}\psi(z,\dot{z}) = \rho_j^n$$

$$\int_{\dot{z}_j^n-\epsilon}^{\dot{z}_j^n+\epsilon} d\dot{z}\mathcal{D}(t,z,\dot{z}) = \hat{D}_j^n$$

$$\int_{-\infty}^{\infty} d\dot{z}\psi(z,\dot{z}) = \rho^n = \Sigma_{j=1}^{2m}\rho_j^n$$

$$\int_{-\infty}^{\infty} d\dot{z}\mathcal{D}(t,z,\dot{z}) = e^{iQ_x t}\rho^n\overline{x} = \Sigma_{j=1}^{2m}\hat{D}_j^n$$

Where $m = N - |N-n|$ with n as the domain slice and N the number of approximating potential wells for a singly peaked potential. The domain slice number n is the number of potential well discontinuities between 0 and z. The domain slice number ranges from 1 to $2N-1$.

$$\frac{\partial\hat{D}_j^n}{\partial t} + \dot{z}_j^n\frac{\partial\hat{D}_j^n}{\partial z} = \frac{i\omega_0}{2Q_x}[\hat{D}_j^n(Q_x^2 + g_1(z,\dot{z}_j^n)) + \rho_j^n e^{iQ_x t} g_2(t,z)] \quad (6)$$

$$\frac{\partial\hat{D}_j^n}{\partial t} + \dot{z}_j^n\frac{\partial\hat{D}_j^n}{\partial z} = \frac{i\omega_0}{2Q_x}[\hat{D}_j^n(C_{sc}\Sigma_{j=1}^{2m}\rho_j^n - \xi(\dot{z}_j^n)) + \rho_j^n(\int_0^z dz' W(z-z')\Sigma_{j=1}^{2m}\hat{D}_j^n - C_{sc}\Sigma_{j=1}^{2m}\hat{D}_j^n)] \quad (7)$$

The wake term still exists, but can be simplified by treating it as assuming the wake is a sum of exponential functions. With this assumption $\int_0^z dz' W(z-z')\Sigma_{j=1}^{2m}\hat{D}_j^n = \Sigma_{k=1}^{\kappa}\hat{F}_k^n$. One can also obtain a set of differential equations describing the wake.

$$\frac{d\hat{F}_k^n}{dz} = w_k\Sigma_{j=1}^{2m}\hat{D}_j^n - \alpha_k\hat{F}_k^n \quad (8)$$

Equation 6 is almost finished but it is still in the form of a linear PDE. These have a family of possible solutions that satisfy the differential equation and boundary conditions. If we assume that $\partial_t\hat{D}_j^n = -i\Delta Q_x\omega_0\hat{D}_j^n$, where ΔQ_x is a parameter that satisfies the boundary conditions, the set of ΔQ_x create a basis for the family of solutions to the PDE. If any of these basis functions have a positive imaginary component, that corresponds to being above the threshold for TMCI.

$$\frac{d\hat{D}_j^n(z)}{dz} = \frac{i\omega_0}{2Q_x\dot{z}_j^n}[\hat{D}_j^n(z)(2Q_x\Delta Q_x - \xi(\dot{z}_j^n) + C_{sc}\Sigma_{n=1}^m\rho_j^n) + \rho_j^n(F_k^n(z) - C_{sc}\Sigma_{n=1}^m\hat{D}_j^n(z))] \quad (9)$$

It should be noted that not all constraints can be satisfied for an arbitrary value of ΔQ_x. If the continuity conditions are not satisfied, then the ΔQ_x is not a basis functions for the PDE solution. Using an incorrect ΔQ_x as a guess, it is possible to converge to a value for the parameter that satisfies boundary conditions by a simple error minimization procedure [1]. By this method, it is possible to obtain the set of basis functions that correspond to the complete family of solutions to the PDE. If any of the solutions are positive imaginary, the problem is above the TMCI threshold.

CODE DEVELOPMENTS

A code is in the process being developed to find the tune shift basis functions for an arbitrary set of square potential wells. Recently, it has observed the onset of the TMCI instability in the constant wake regime in a single square potential well and is now in the process of benchmarking against results from other simulations. Constant wake simulations are within 1% (Fig. 13 [3])of previously quoted figures. Once the efficacy of the code is proven, the method will be adapted to multiple potential wells.

CONCLUSION

The method proposed for a multiple square well model offers the capability to include collective effects in a quickly converging semianalytic model that should be able to predict TMCI thresholds more accurately than previously possible with similar models. A code to implement this method to multiple wells is in progress.

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