Kinetic Limits to Average Power in Plasma Wakefield Accelerators

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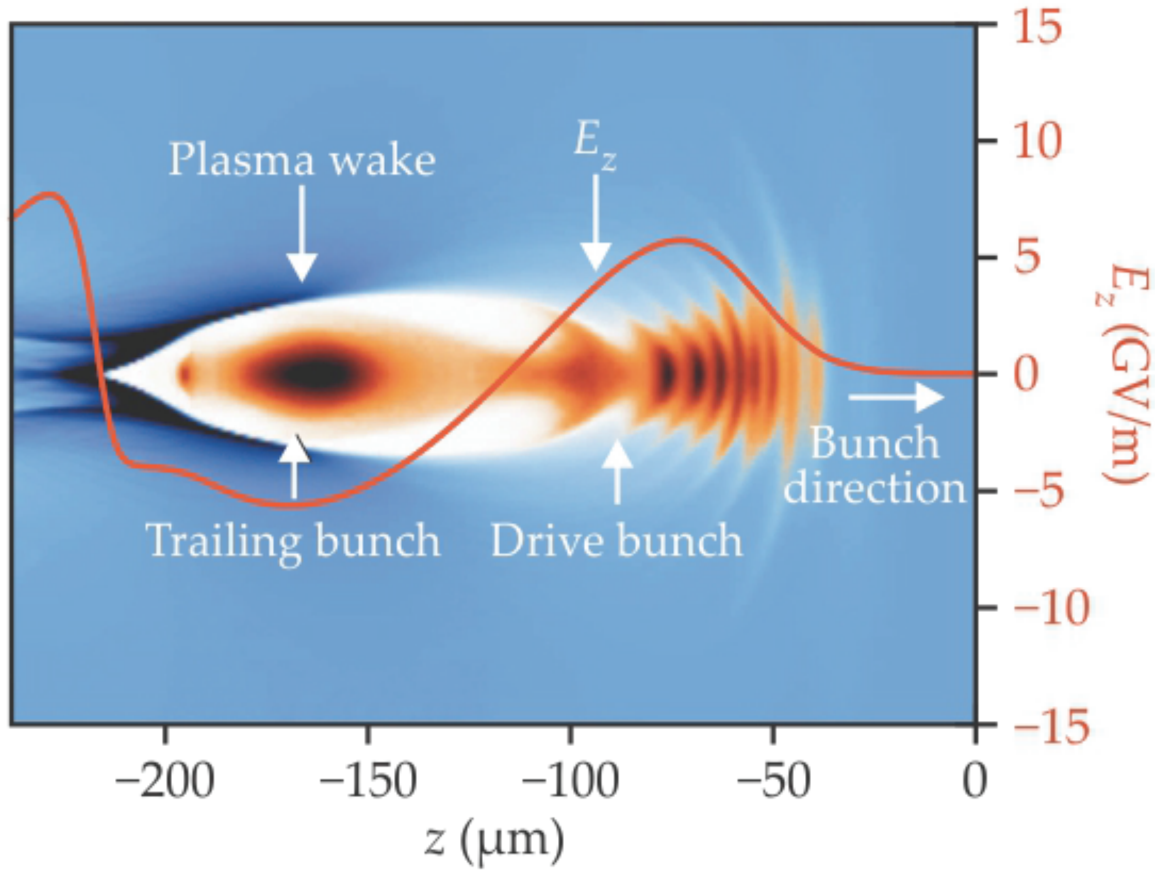
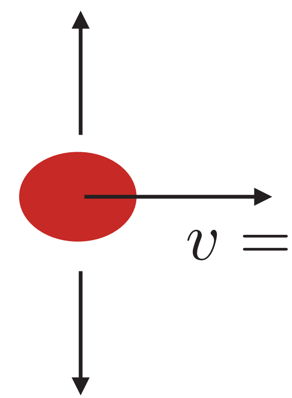
The Problem: A Beam-Driven Plasma Wakefield Accelerator



M. Litos, et al., "High-efficiency acceleration of an electron beam in a plasma wakefield accelerator", Nature **515**, 92–95 (2014).

A Model for Beam-Driven Wakes

$$H = \underbrace{\frac{p^2}{2mc} + \frac{1}{4} m c k_p^2 r^2}_{H_0} + \underbrace{\phi_e(r, \xi)}_{H_1}$$



$$v = c$$

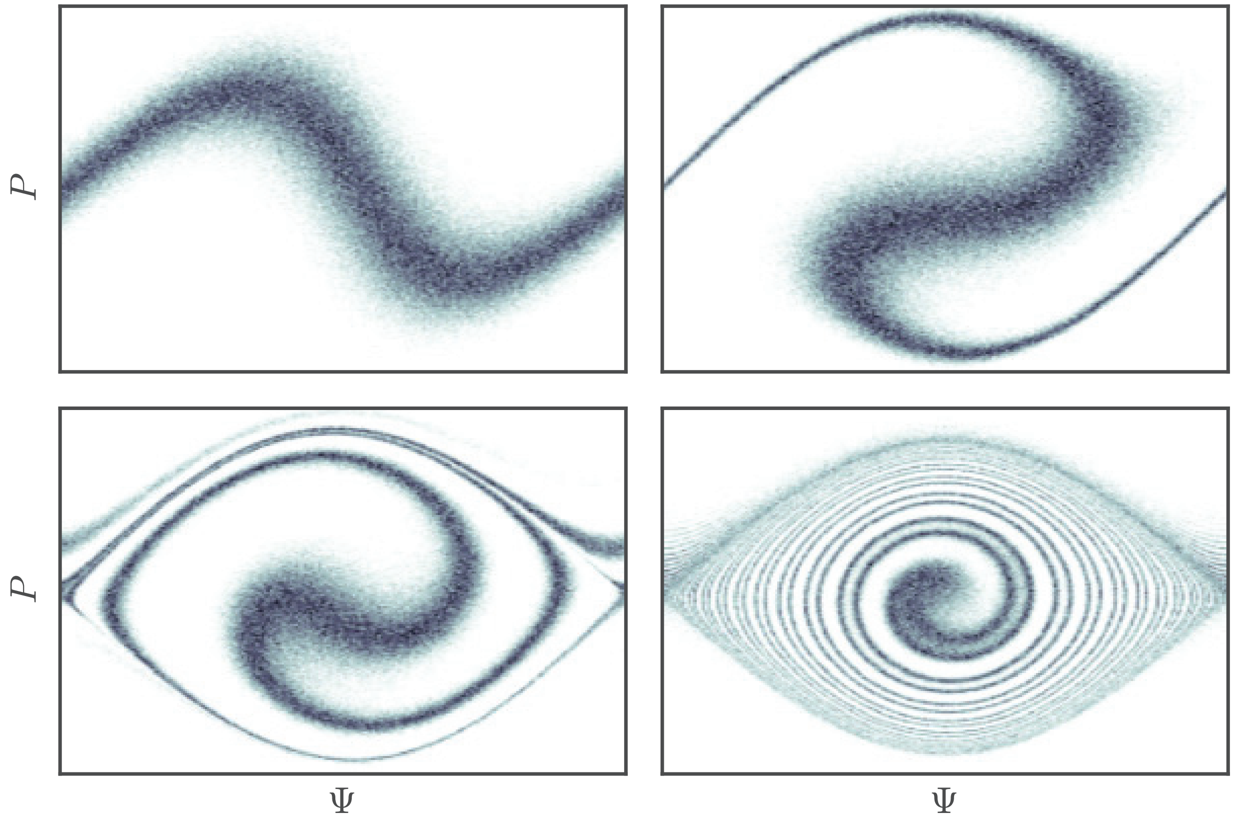
$$\xi = z - ct$$

$$E_r(r) = \frac{Ne}{4\pi\sigma_b^2 \ell_b} \begin{cases} \frac{r}{\sigma_b} & r < \sigma_b \\ \frac{\sigma_b}{r} & r \geq \sigma_b \end{cases}$$

Solving the Vlasov Equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + q \left(\vec{E}[f] + \frac{\vec{v}}{c} \times \vec{B}[f] \right) \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

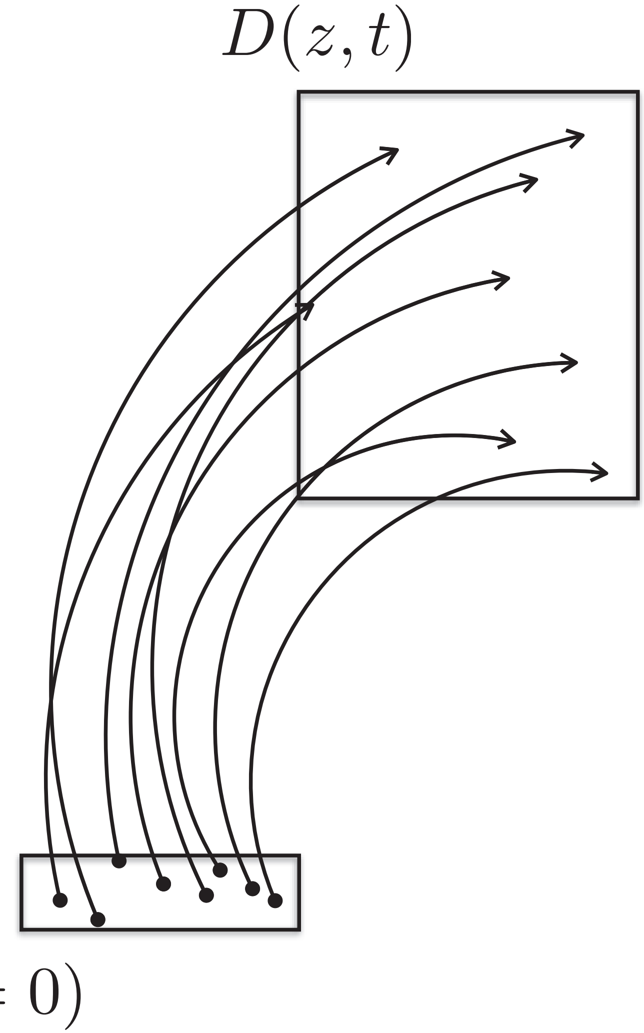
$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} + q \left(\vec{E}[f_1] + \frac{\vec{v}}{c} \times \vec{B}[f_1] \right) \cdot \frac{\partial f_0}{\partial \vec{v}} + q \left(\vec{E}[f_1] + \frac{\vec{v}}{c} \times \vec{B}[f_1] \right) \cdot \frac{\partial f_1}{\partial \vec{v}} = 0$$



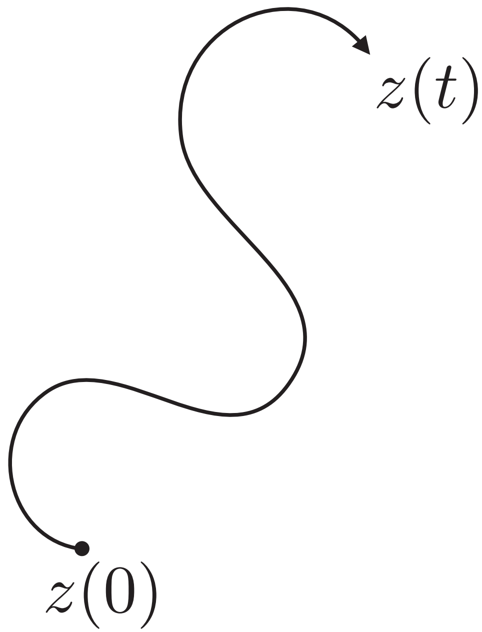
Phase space filamentation makes derivatives large after long times

The Vlasov Equation & Hamiltonian Flows

$$D(z,t) = D(\mathscr{M}_{-t} \circ z_0, t=0)$$



Hamiltonian Flows & Symplectic Maps



$$\dot{z} = -\{H, z\}$$

$$z(t) = \mathscr{M}_t \circ z(0)$$

$$\{H, *\} \leftrightarrow :H:$$

$$\dot{\mathscr{M}} = \mathscr{M} :-H:$$

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

$$\mathscr{M} \doteq \begin{pmatrix} \cos\omega t & \omega^{-1}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}$$

Split Operator Hamiltonian Perturbation Theory

$$H = H_0 + \epsilon H_1$$

$$\mathscr{M} = \mathscr{M}_I \mathscr{M}_0 \qquad \dot{\mathscr{M}}_0 = \mathscr{M}_0 :-H_0:$$

$$\dot{\mathscr{M}}_I = \epsilon \mathscr{M}_I \underbrace{\left(\mathscr{M}_0 :-H_1: \mathscr{M}_0^{-1}\right)}_{:-H_I:}$$

$$H_I(z,t) = H_1(\mathscr{M}_0 \circ z, t)$$

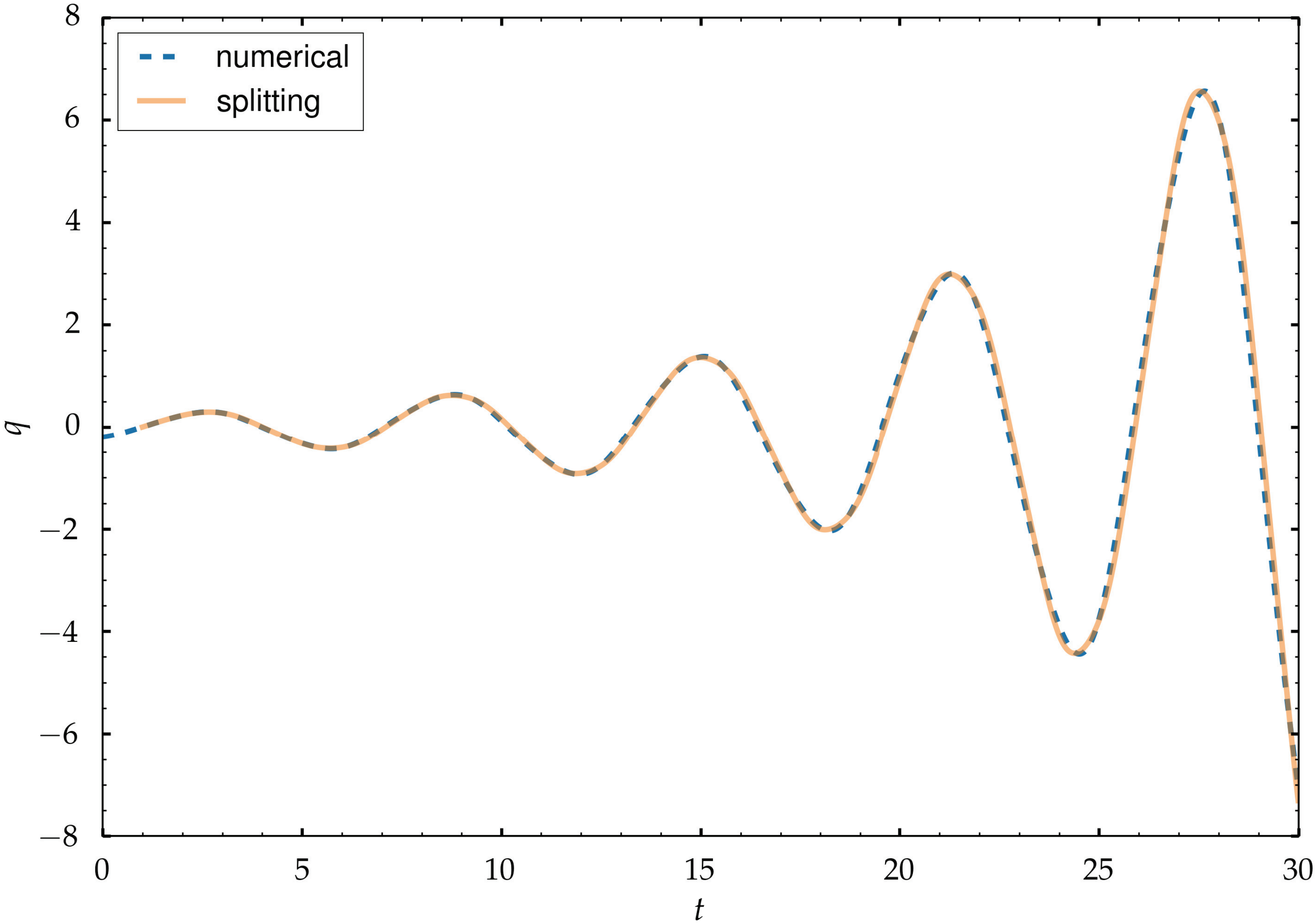
$$\mathscr{M}_I(t) \approx \exp\left\{:-\epsilon \int_0^t d\sigma H_I(z,\sigma): + \mathcal{O}\left((\epsilon t)^2\right)\right\}$$

$$H = \underbrace{\frac{1}{2}\left(p^2 + \omega^2 q^2\right)}_{H_0} + \epsilon \underbrace{\frac{1}{2}\omega^2 \cos(2\omega t) q^2}_{H_1}$$

$$H_I = \frac{1}{2}\left[\cos(2\omega t)\left(p^2\cos^2(\omega t) + \omega^2 q^2 \sin^2(\omega t)\right) + 2\omega pq \cos(\omega t)\cos(2\omega t)\sin(\omega t)\right]$$

$$\mathscr{M}_I \approx \exp\left\{\epsilon \int_0^t ds \;\cos(2\omega s) \begin{pmatrix} \sin(\omega s)\cos(\omega s) & \sin^2(\omega s)/\omega \\ -\omega^2\cos^2(\omega s) & -\omega\sin(\omega s)\cos(\omega s) \end{pmatrix} + \mathcal{O}(t^2)\right\}$$

Hill's Equation: A Quick Example



Hill’s Equation: A Quick Example

Application to the Nonlinear Vlasov Equation

$$H = H_0(p,q) + \epsilon \mathcal{H}_1(p,q;f) \qquad f(z,t) = f\left(\left[\mathscr{M}_0^{-1}\mathscr{M}_I^{-1}\right] \circ z_0, 0\right)$$

$$\dot{\mathscr{M}}_I = \mathscr{M}_I : -\epsilon \int dz' dt' \mathcal{G}(\mathscr{M}_t^0 z^i, t; z', t') f(\mathscr{M}_{-t} z'^i, 0) : .$$

$$\mathscr{M}_I = \exp\left[:\Omega^{[N]}(t):\right]$$

$$\Omega^{[1]}(t) = -\epsilon \int_0^t d\tau \int dz' dt' \mathcal{G}(\mathscr{M}_\tau^0 z^i, t; z', t') f(\mathscr{M}_{-t'}^0 z'^i, 0)$$

$$\Omega^{[N]}(t) = -\epsilon \sum_{n=0}^{N-2} \frac{B_n}{n!} \int_0^t d\tau \; :\Omega^{[N-1]}(\tau):^n \int dz' dt' \mathcal{G}(\mathscr{M}_\tau^0 z, \tau; z', t') f(e^{:\Omega^{[N-1]}(-t'):} \mathscr{M}_{-t'}^0 z', 0), N \geq 2$$

S. Webb, "A Hamiltonian perturbation theory for the nonlinear Vlasov equation", J. Math. Phys. **57**, 042905 (2016)

Short Drive Beam Approximation

beam parameter

$$b_0 = \frac{Ne^2}{2\pi\sigma_b^2 m_e c^2}$$

short beam transverse momentum kick

$$P(R) = mcb_0 \begin{cases} \frac{R}{\sigma_b} & R < \sigma_b \\ \frac{\sigma_b}{R} & R \geq \sigma_b \end{cases}$$

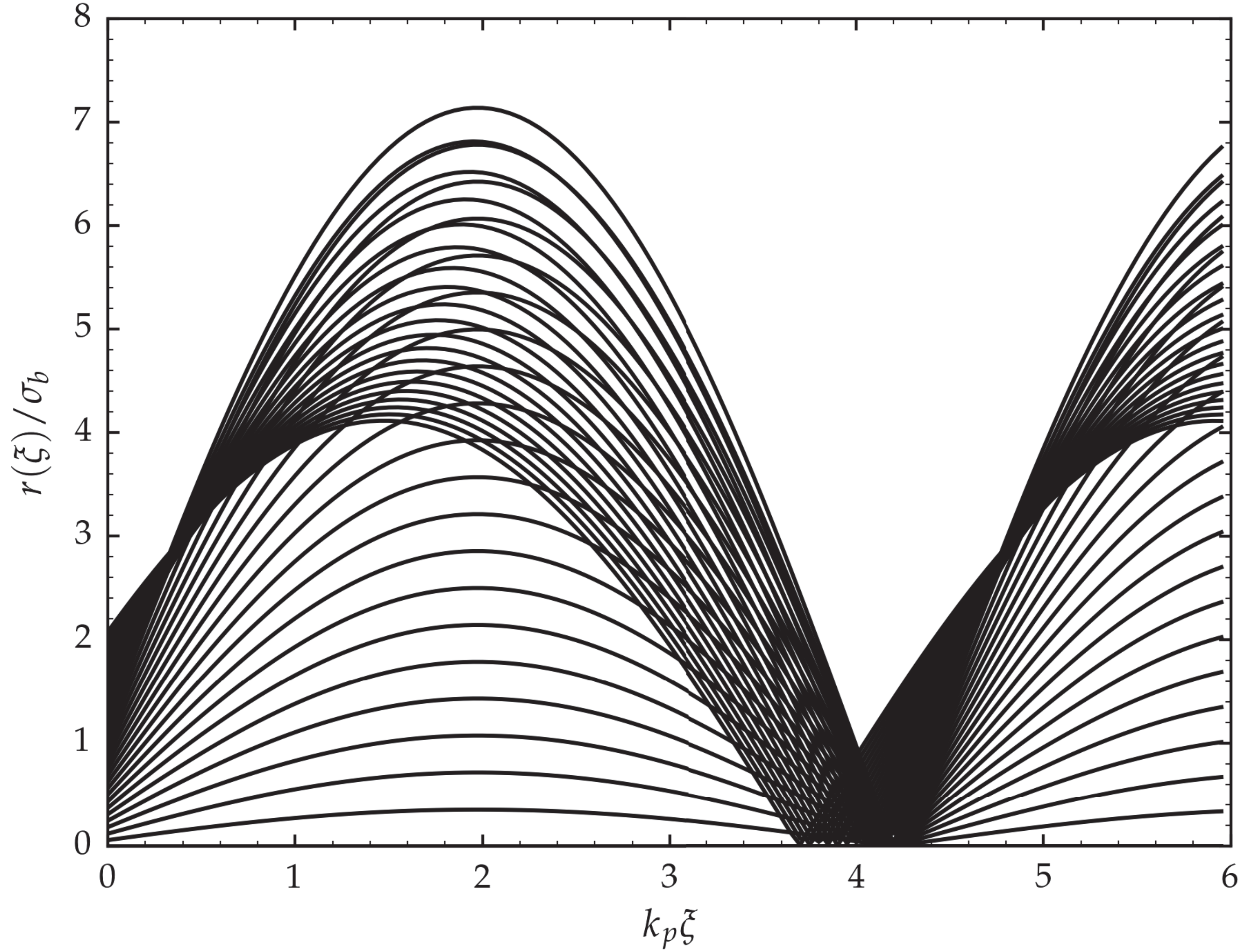
non-relativistic limit

$$b_0 \ll 1$$

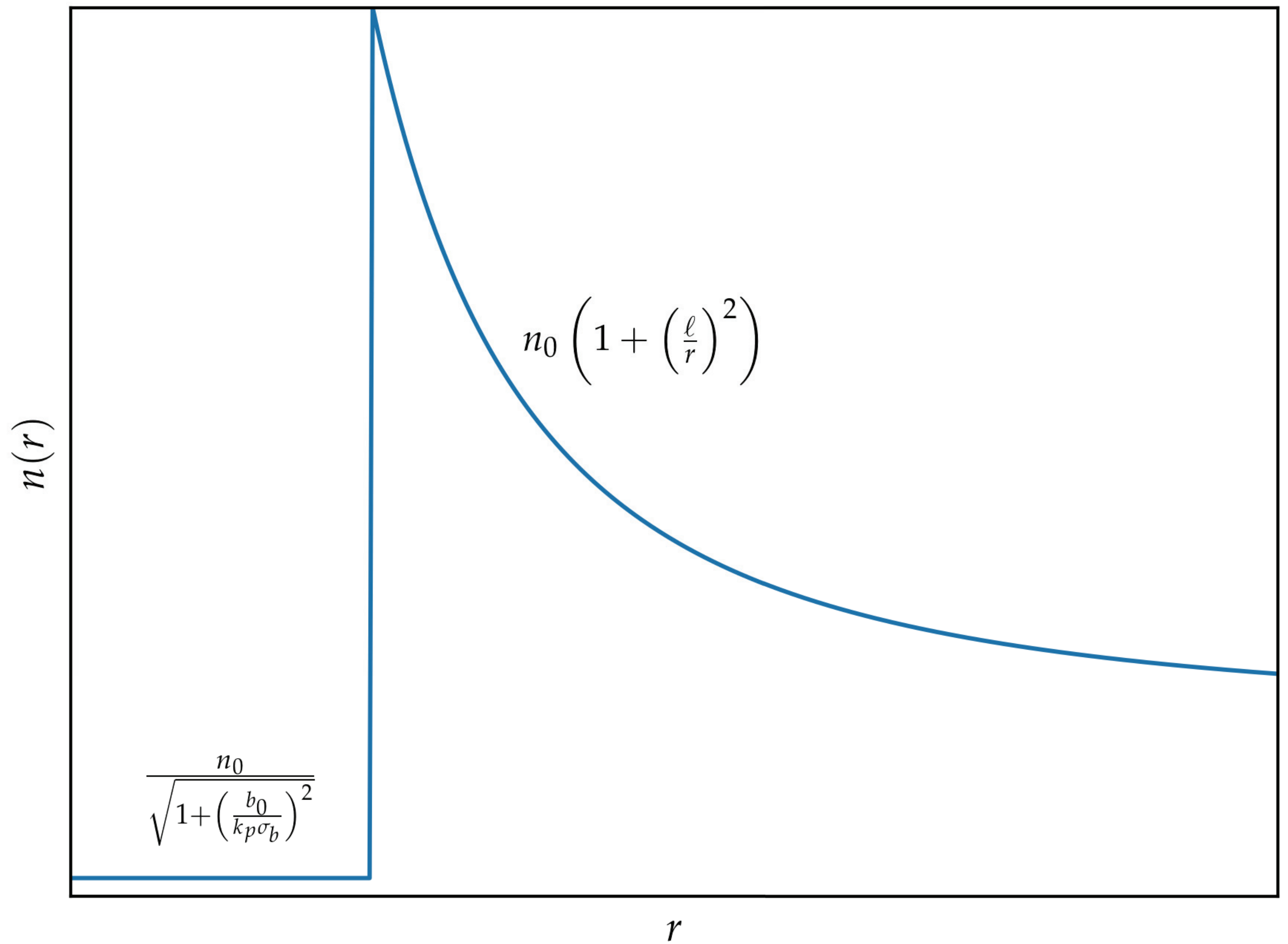
initial distribution

$$f\left(p_0, r_0, \xi = 0\right) = \delta\left(p_0 - P(r_0)\right)$$

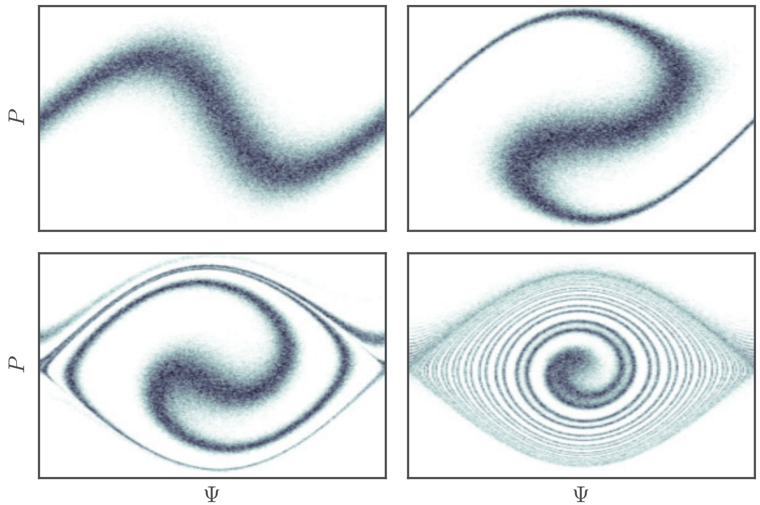
Zeroth Order Theory Predicts Wake Shape



Zeroth Order Theory Predicts Wake Shape



Analytic Estimate for the Decoherence Time



$$\omega(\mathcal{E})^{-1} = \oint dq \sqrt{\frac{2M}{\mathcal{E} - \mathcal{V}(q)}}$$

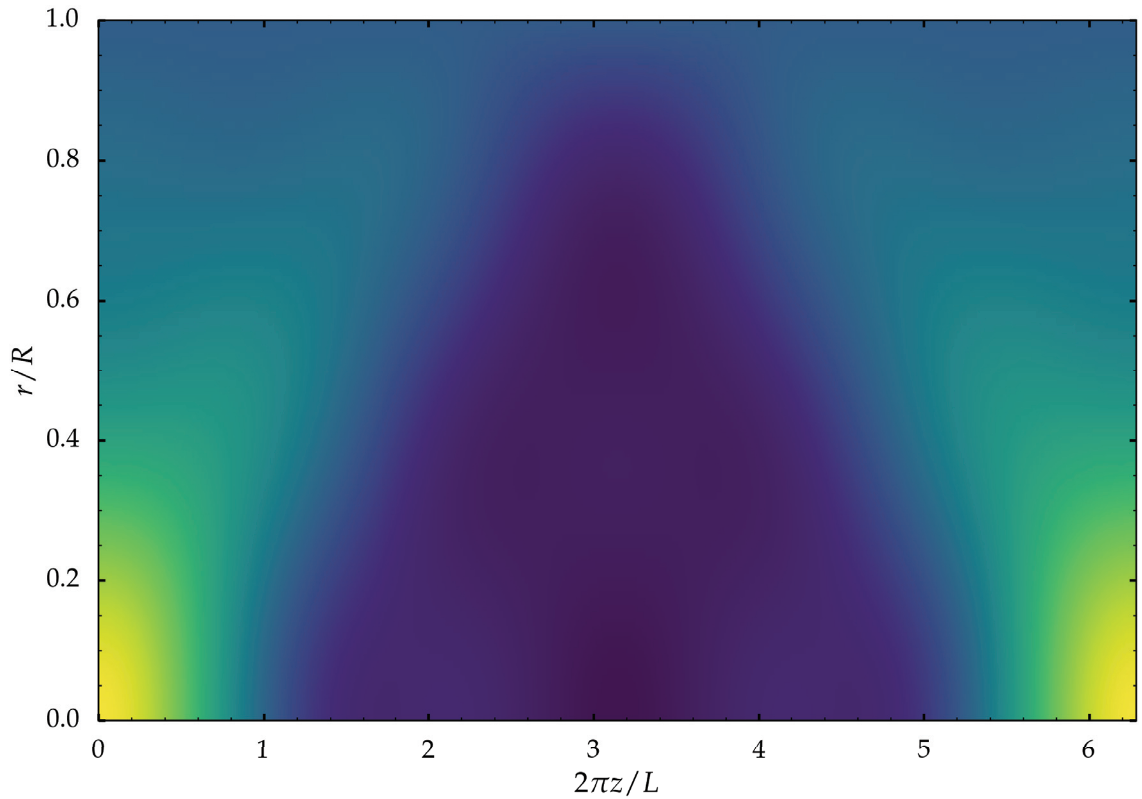
$$\tau \sim \left\langle (\omega^{-1})^2 \right\rangle_{\mathcal{E}} - \left\langle \omega^{-1} \right\rangle_{\mathcal{E}}^2$$

Numerical Validation — A Symplectic Particle-in-Mode Algorithm

$$\mathcal{L} = \int d\mathbf{x}_0 d\mathbf{v}_0 \left\{ -mc\sqrt{1 - \dot{r}^2 - \dot{z}^2 - r^2\dot{\theta}^2} + \frac{q}{c}\dot{r}A_r + \frac{q}{c}\dot{z}A_z \right\} \Psi(\mathbf{x}_0, \mathbf{v}_0) + \frac{1}{8\pi c}\int d\mathbf{x} \left(\frac{\partial \mathbf{A}}{\partial \tau}\right)^2 - (\nabla \times \mathbf{A})^2.$$

Eigenmodes of a Cylindrical Cavity

$$\mathbf{A} = \sum_{m,n} a_{m,n}(\tau) \begin{pmatrix} J_1\left(x_m \frac{r}{R}\right) \sin\left(\frac{2n\pi z}{L}\right) \\ J_0\left(x_m \frac{r}{R}\right) \cos\left(\frac{2n\pi z}{L}\right) \end{pmatrix}$$



Particle Shape Functions

$$\Psi(\mathbf{x}, \mathbf{v}) = \sum_{j=1}^{N_{macro.}} w_j \frac{1}{r} \Lambda_r(r - r_j) \Lambda_z(z - z_j)$$

$$\delta(v_r - \dot{r}_j)\delta(v_z - \dot{z}_j) \exp\left(-m\frac{r^2\dot{\theta}^2}{2T}\right)$$

$$
\begin{aligned}
\mathcal{L} = \sum_j \Bigg\{ & -mcw_j \sqrt{1 - \dot{r}_j^2 - \dot{z}_j^2 - \frac{1}{r_j^2}\frac{\ell_j^2}{mc}} \\
& + 2\pi w_j \frac{q}{c} \left(\dot{r}_j I_r(\mathbf{x}_j) + \dot{z}_j I_z(\mathbf{x}_j) \right) \Bigg\} \\
& + \frac{1}{4c} \sum_{m,n} \left(\dot{a}_{m,n} \right)^2 C_{m,n} - \frac{1}{L_{m,n}} \left(a_{m,n} \right)^2 .
\end{aligned}
$$

The Spatially Discretized Relativistic Lagrangian

$$\mathcal{H} = \underbrace{\sum_j \sqrt{\left(\mathbf{p}_j - 2\pi\frac{q}{c}\sum_{m,n}\mathcal{Q}_{m,n}\sqrt{\frac{2c}{C_{m,n}}}\mathbf{F}_{m,n}(\mathbf{q}_j)\right)^2 + \frac{\ell_j^2}{mcr_j^2} + m^2c^2} +}_{\mathcal{H}_{p-c}}$$

$$\underbrace{\frac{1}{2}\sum_{m,n}(\mathcal{P}_{m,n})^2 + \frac{1}{c^2}\frac{1}{C_{m,n}L_{m,n}}(\mathcal{Q}_{m,n})^2}_{\mathcal{H}_f}$$

The Spatially Discretized Relativistic Hamiltonian

$$\mathcal{N} = e^{-:\mathcal{H}_f:h/2} e^{-:\mathcal{H}_{p-c}:h} e^{-:\mathcal{H}_f:h/2}$$

Numerical Map

$$\mathcal{N} = e^{-:\mathcal{H}_f:h/2} e^{-:\mathcal{H}_{p-c}:h} e^{-:\mathcal{H}_f:h/2}$$

$$e^{-:\mathcal{H}_f:h/2} \circ \begin{pmatrix} \mathcal{P}_{m,n} \\ \mathcal{Q}_{m,n} \end{pmatrix} = \begin{pmatrix} \cos \Omega_{m,n} \frac{h}{2} & \Omega_{m,n} \sin \Omega_{m,n} \frac{h}{2} \\ -\frac{1}{\Omega_{m,n}} \sin \Omega_{m,n} \frac{h}{2} & \cos \Omega_{m,n} \frac{h}{2} \end{pmatrix} \begin{pmatrix} \mathcal{P}_{m,n} \\ \mathcal{Q}_{m,n} \end{pmatrix}$$

Field Map

$$\mathscr{N} = e^{-:\mathcal{H}_f:h/2} e^{-:\mathcal{H}_{p-c}:h} e^{-:\mathcal{H}_f:h/2}$$

$$\dot{\mathscr{M}}_{p-c} = \mathscr{M}_{p-c} : -\left\{ \sum_j \sqrt{\left(\mathbf{p}_j - 2\pi\frac{q}{c} w_j \sum_{m,n} \mathcal{Q}_{m,n} \sqrt{\frac{2c}{C_{m,n}}} \mathbf{F}_{m,n}(\mathbf{q}_j) C_{m,n}\right)^2 + \frac{\ell_j^2}{r_j^2} + m^2c^2} \right\} :$$

Particle-Coupling Map

Proper Time vs. Lab Time Tracking

$$ds = d\tau/\gamma$$

$$L_s = -\frac{1}{2}mcU^iU_i - \frac{q}{c}U^iA_i \qquad U = (\gamma, \gamma\mathbf{x}')$$

$$K_s = -\frac{(P^i - eA^i)(P_i - eA_i)}{2mc}$$

Y.K. Wu, É. Forest, and D. S. Robin, “Explicit symplectic integrator for s-dependent static magnetic fields”, Phys. Rev. E **68**, 046502 (2003).

Proper Time vs. Lab Time Tracking

$$\gamma_j mc = \sqrt{\left(\mathbf{p}_j - \frac{q}{c} w_j 2\pi \sum_{m,n} \mathcal{Q}_{m,n} \sqrt{\frac{2c}{C_{m,n}}} \mathbf{F}_{m,n}(\mathbf{q}_j)\right)^2 + \frac{\ell_j^2}{r_j^2} + m^2c^2}$$

$$\overline{\mathcal{H}}_{p-c} = \frac{\left(\mathbf{p}_j - \frac{q}{c} w_j 2\pi \sum_{m,n} \mathcal{Q}_{m,n} \sqrt{\frac{2c}{C_{m,n}}} \mathbf{F}_{m,n}(\mathbf{q}_j)\right)^2 + \frac{\ell_j^2}{r_j^2}}{2mc\gamma_j}$$

Similarity Transform on the Magnetic Map

$$e^{-:(p_i-a_i(\mathbf{q}))^2:t} = e^{:-\int a_i(\mathbf{q})dq_i:}e^{-:p_i^2:t}e^{-:-\int a_i(\mathbf{q})dq_i:}$$

$$\mathscr{M}^{(r)}(h) = \exp\left\{-\sum_j : \frac{p_j^{(r)} - \frac{q}{c}w_j 2\pi \sum_{m,n} \mathcal{Q}_{m,n}\sqrt{\frac{2}{C_{m,n}}}F_{m,n}^{(r)}(\mathbf{q}_j)}{2\gamma_j mc} : h\right\}$$

$$\mathscr{M}^{(r)}(h) = \mathscr{A}^{(r)}\mathscr{D}^{(r)}\left[\mathscr{A}^{(r)}\right]^{-1}$$

$$\mathscr{D}^{(r)} = e^{-\sum_j :p_j^{(r)\,2}/2\gamma_j mc:\,h}$$

$$\mathscr{A}^{(r)} = e^{-\sum_j \frac{q}{c}w_j 2\pi :\int dr_j \sum_{m,n}\mathcal{Q}_{m,n}\sqrt{\frac{2}{C_{m,n}}}F_{m,n}^{(r)}(\mathbf{q}_j):}$$

Y.K. Wu, É. Forest, and D. S. Robin, “Explicit symplectic integrator for s-dependent static magnetic fields”, Phys. Rev. E **68**, 046502 (2003).

The Full Integrator

$$\mathscr{N} = \mathscr{M}_f(h/2)\,\underbrace{\mathscr{M}^{(\ell)}(h/2)\mathscr{A}^{(r)}\mathscr{D}^{(r)}(h/2)\left[\mathscr{A}^{(r)}\right]^{-1}\mathscr{A}^{(z)}\mathscr{D}^{(z)}\left[\mathscr{A}^{(z)}\right]^{-1}\mathscr{A}^{(r)}\mathscr{D}^{(r)}(h/2)\left[\mathscr{A}^{(r)}\right]^{-1}\mathscr{M}^{(\ell)}(h/2)}_{e^{-:\mathscr{H}_{p-c}:h+\mathcal{O}(h^3)}}\,\mathscr{M}_f(h/2)$$

13 map operations
6 distinct maps

includes angular momentum and self-consistent deposition/interpolation

a few unusual properties...

Thank you!

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