Fermilab

SCHARGEV 1.0 - Strong Space Charge Vlasov Solver [THPOA30]

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SCHARGEV 1.0

The space charge (SC) is known to be one of the major limitations for the collective transverse beam stability. When space charge is strong (SSC), i.e. space charge tune shift $\gg$ synchrotron tune Q_s, the problem allows an exact analytical solution. For that practically important case we present a fast and effective Vlasov solver SCHARGEV 1.0 (Space CHARGE Vlasov) which calculates a complete eigensystem (spatial shapes of modes and frequency spectra) and therefore provides the growth rates and the thresholds of instabilities. SCHARGEV 1.0 includes driving and detuning wake forces, and, any feedback system (damper). In the next version we will include coupled bunch interaction and Landau damping. Numerical examples for FermiLab Recycler and CERN SPS are presented.

1. Strong Space Charge Theory [1–3]

SSC harmonics for a bunch with longitudinal distribution function $f(\tau, \mathrm{v})$, where τ is measured in radians, satisfy

$$\begin{cases} \dfrac{1}{Q_{\text{eff}}(\tau)}\dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\dfrac{\mathrm{d}\,Y(\tau)}{\mathrm{d}\tau}\right) + \nu\, Y(\tau) = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\Big|_{\tau=\pm\infty} = 0 \end{cases} \qquad \int \rho(\tau)\, Y_l(\tau)\, Y_m(\tau)\,\mathrm{d}\,\tau = \delta_{lm} \qquad \int \rho(\tau)\,\mathrm{d}\,\tau = 1$$

$$Q_{\text{eff}}(\tau) = Q_{\text{eff}}(0)\,\frac{\rho(\tau)}{\rho(0)}$$ **— effective space charge tune shift**

$$\rho(\tau) = \int f(\tau,\mathrm{v})\,\mathrm{d}\,\mathrm{v}$$ **— normalized line density**

$$u^2(\tau) = \int f(\tau,\mathrm{v})\,\mathrm{v}^2\,\mathrm{d}\,\mathrm{v} \Big/ \rho(\tau)$$ **— average square of particle longitudinal velocity**

The modified dynamic equation including the wake and the damper is

$$\frac{1}{Q_{\text{eff}}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\frac{\mathrm{d}\,\mathcal{Y}(\tau)}{\mathrm{d}\tau}\right) + \omega\,\mathcal{Y}(\tau) = \left[\varkappa\left(\widehat{\mathrm{W}} + \widehat{\mathrm{D}}\right) - i\,g\,e^{i\,\psi}\,\widehat{\mathrm{G}}\right]\mathcal{Y}(\tau)\,, \qquad \varkappa = N_{\mathrm{b}}\,\frac{r_0 R_0}{4\,\pi\,\gamma\,\beta^2\,Q_\beta}$$

where the operators of wake forces are defined in terms of driving and detuning wakes, and damper is defined through the pickup $P(\tau)$ and kicker $K(\tau)$ functions

$$\widehat{\mathrm{W}}\,Y(\tau) = \int_\tau^\infty W(\tau-\sigma)\,\rho(\sigma)\,Y(\sigma)\,e^{i\,\zeta(\tau-\sigma)}\,\mathrm{d}\,\sigma\,,$$

$$\widehat{\mathrm{D}}\,Y(\tau) = Y(\tau)\int_\tau^\infty D(\tau-\sigma)\,\rho(\sigma)\,\mathrm{d}\,\sigma\,,$$

$$\widehat{\mathrm{G}}\,Y(\tau) = K(\tau)\int_{-\infty}^\infty P(\sigma)\,\rho(\sigma)\,Y(\sigma)\,e^{i\,\zeta(\tau-\sigma)}\,\mathrm{d}\,\sigma.$$

- r_0 **— classical radius of the beam particle**
- R_0 **— average accelerator ring radius**
- N_{b} **— number of particles per bunch**
- Q_β **— bare betatron tune**
- $\zeta = -\xi/\eta$ **— where ξ is chromaticity and $\eta = \gamma_t^{-2} - \gamma^{-2}$ is slippage factor**
- g **and** ψ **— dimensionless gain and damper's phase respectively**

Expansion over SSC harmonics $\mathcal{Y}_k(\tau) = \sum_{i=0}^\infty \mathbf{C}_i^{(k)} Y_i(\tau)$ gives the eigenvalue problem

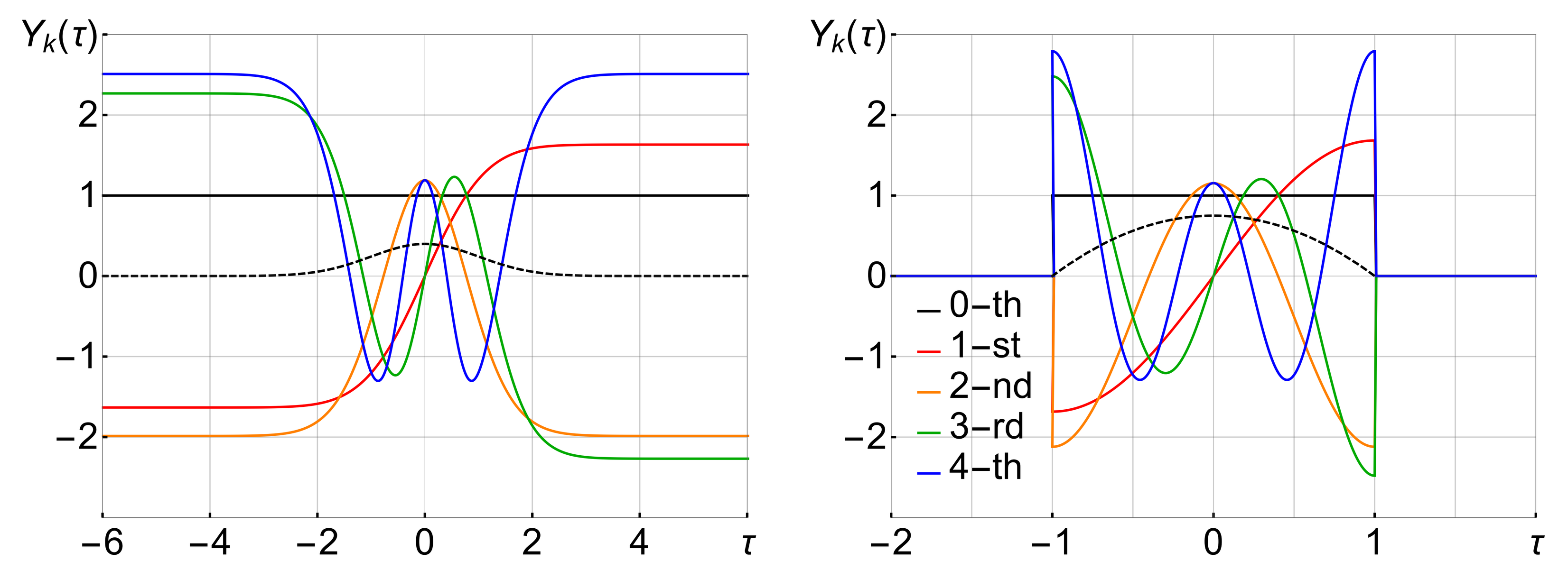
$$\mathbf{M}\cdot\mathbf{C}^{(k)} = \omega_k \mathbf{C}^{(k)} \qquad \mathbf{M}_{lm} = \nu_l\,\delta_{lm} + \varkappa\left[\widehat{\mathrm{W}}_{lm}(\zeta) + \widehat{\mathrm{D}}_{lm}\right] - i\,g\,e^{i\,\psi}\,\widehat{\mathrm{G}}_{lm}(\zeta)$$

$$\widehat{\mathrm{W}}_{lm} = \int_{-\infty}^\infty\int_\tau^\infty \mathrm{d}\,\sigma\,\mathrm{d}\,\tau\,W(\tau-\sigma)\,\rho(\tau)\,\rho(\sigma)\,Y_l(\tau)\,Y_m(\sigma)\,e^{i\,\zeta(\tau-\sigma)}\,,$$

$$\widehat{\mathrm{D}}_{lm} = \int_{-\infty}^\infty\int_\tau^\infty \mathrm{d}\,\sigma\,\mathrm{d}\,\tau\,D(\tau-\sigma)\,\rho(\tau)\,\rho(\sigma)\,Y_l(\tau)\,Y_m(\tau)\,,$$

$$\widehat{\mathrm{G}}_{lm} = \mathrm{K}_l(\zeta)\,\mathrm{P}_m^*(\zeta) = \int_{-\infty}^\infty \mathrm{d}\,\tau\,K(\tau)\,\rho(\tau)\,Y_l(\tau)\,e^{i\,\zeta\,\tau}\left(\int_{-\infty}^\infty \mathrm{d}\,\tau\,P(\tau)\,\rho(\tau)\,Y_m(\tau)\,e^{i\,\zeta\,\tau}\right)^*.$$

2.1 SSC Harmonics



Longitudinal phase-space models: Gaussian and Hoffman-Pedersen distributions

$$f^{(\mathrm{G})}(\tau,\mathrm{v}) = \frac{1}{2\,\pi\,a\,b}\exp\left[-\frac{\tau^2}{2\,a^2} - \frac{\mathrm{v}^2}{2\,b^2}\right] \qquad f^{(\mathrm{HP})}(\tau,\mathrm{v}) = \frac{3}{2\,\pi\,a\,b}\sqrt{1 - \frac{\tau^2}{a^2} - \frac{\mathrm{v}^2}{b^2}}$$

Line density $\rho(\tau)$ and average square of the particle longitudinal velocity $u^2(\tau)$

$$\rho(\tau) = \frac{1}{\sqrt{2\,\pi}\,a}\exp\left[-\frac{\tau^2}{2\,a^2}\right] \qquad \rho(\tau) = \frac{3}{4\,a}\left(1 - \frac{\tau^2}{a^2}\right)$$

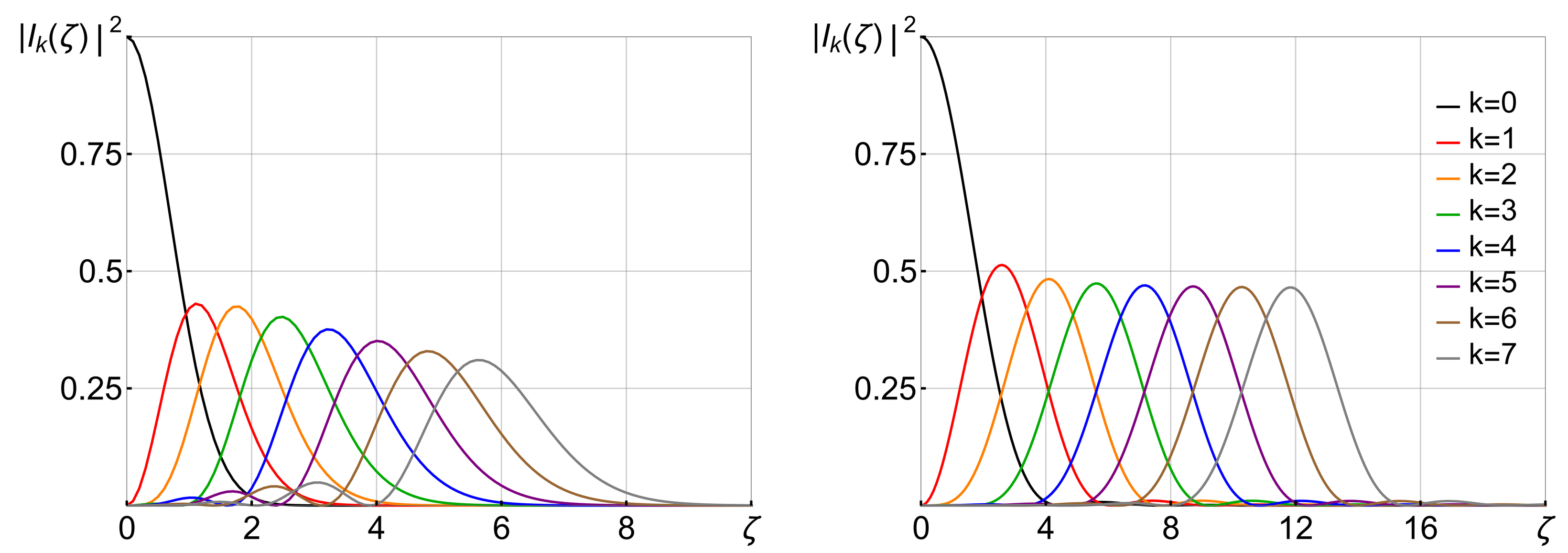
$$u^2(\tau) = b^2 \qquad u^2(\tau) = \frac{b^2}{4}\left(1 - \frac{\tau^2}{a^2}\right)$$

Eigenproblems for SSC harmonics (τ in units of $[a]$, ν in units of $[Q_s^2/Q_{\text{eff}}(0)]$)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left[\dfrac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\right] + \nu\,e^{-\tau^2/2}\,Y(\tau) = 0 \\ Y'(\pm\infty) = 0 \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left[(1-\tau^2)\dfrac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\right] + 4\,\nu\,(1-\tau^2)\,Y(\tau) = 0 \\ Y'(\pm 1) = 0 \end{cases}$$

k	0	1	2	3	4	5
$\nu_k^{(\mathrm{G})}$	0	1.342	4.325	8.898	15.053	22.787
$\nu_k^{(\mathrm{HP})}$	0	1.156	3.591	7.271	12.191	18.347

2.2 Bunch Dipole Moments and Flat Bunch-by-Bunch Damper



Dipole moments characterize contribution of the specific harmonic to the motion of a total center of mass. As functions of head-tail phase they are

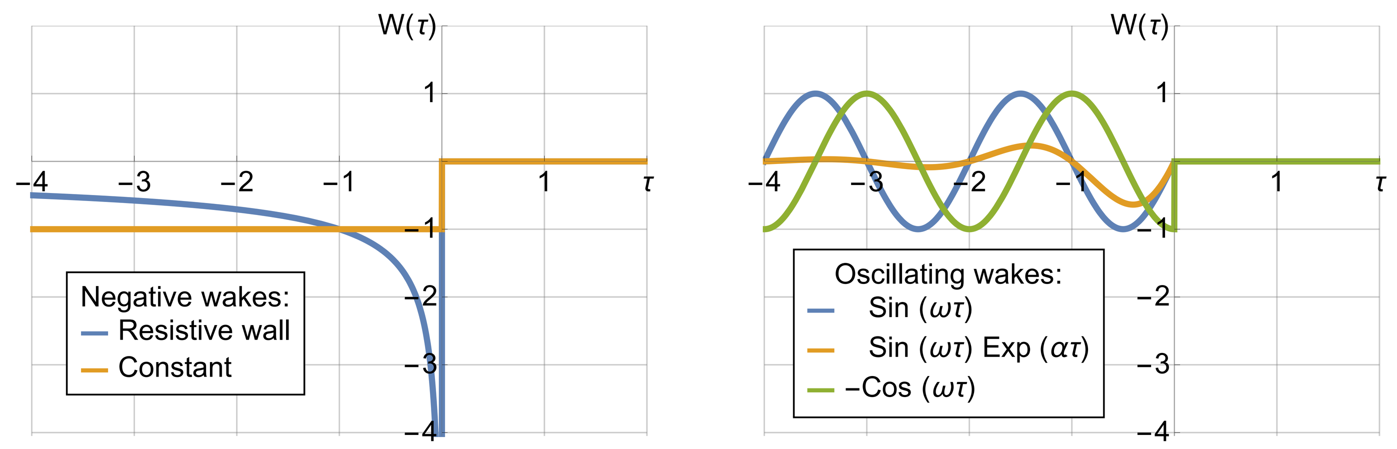
$$I_k(\zeta) = \int_{-\infty}^\infty \rho(\tau)\,Y_k(\tau)\,e^{i\,\zeta\,\tau}\,\mathrm{d}\,\tau \qquad I_k^*(\zeta) = I_k(-\zeta) = (-1)^k I_k(\zeta)$$

When damper sees the center of mass of an individual bunch and applies a dipole kick uniformly along its length, it can be taken into account as a matrix of direct product of dipole moments

$$\widehat{\mathrm{G}}_{lm} = \int_{-\infty}^\infty\int_{-\infty}^\infty \mathrm{d}\,\tau\,\mathrm{d}\,\sigma\,\rho(\tau)\rho(\sigma)Y_l(\tau)Y_m(\sigma)\,e^{i\,\zeta(\tau-\sigma)} = I_l(\zeta)I_m^*(\zeta) = (-1)^m I_l(\zeta)I_m(\zeta).$$

2.3 Wake Forces

SCHARGEV's default library includes resistive wall and broad-band resonator wake fields, and, model constant and oscillating wakes.



Arbitrary wake field can be used for construction of matrix elements

$$\widehat{\mathrm{W}}_{lm}(\zeta) = i\,(-1)^{m+1}\int_{-\infty}^\infty \mathrm{Z}^\perp(\omega-\zeta)I_l(\omega)I_m(\omega)\frac{\mathrm{d}\omega}{2\pi}$$

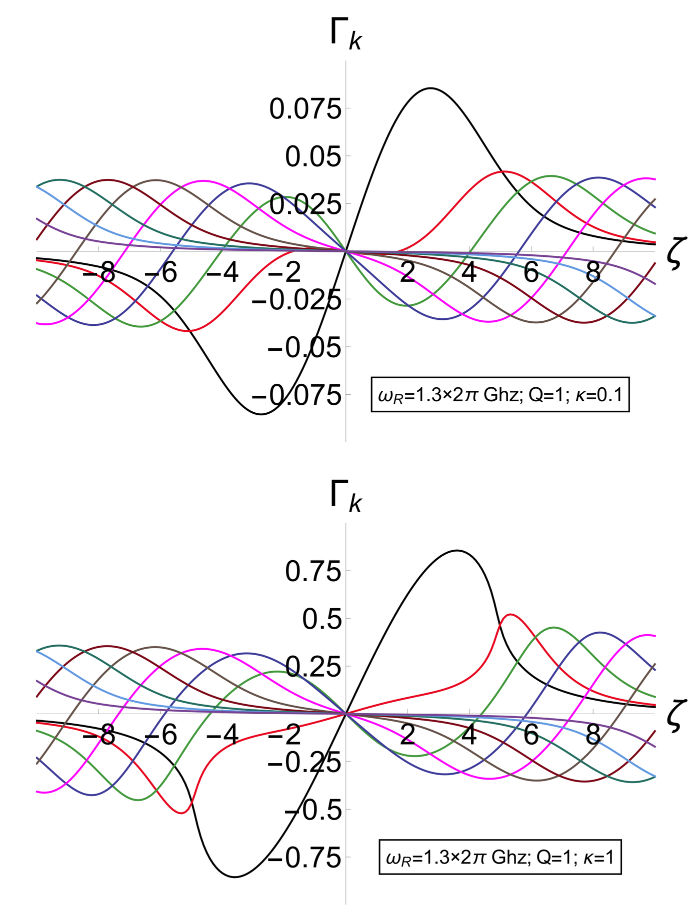
$$\widehat{\mathrm{D}}_{lm} = \int_{-\infty}^\infty F(\tau)\rho(\tau)Y_l(\tau)Y_m(\tau)\,\mathrm{d}\tau,$$ **where**

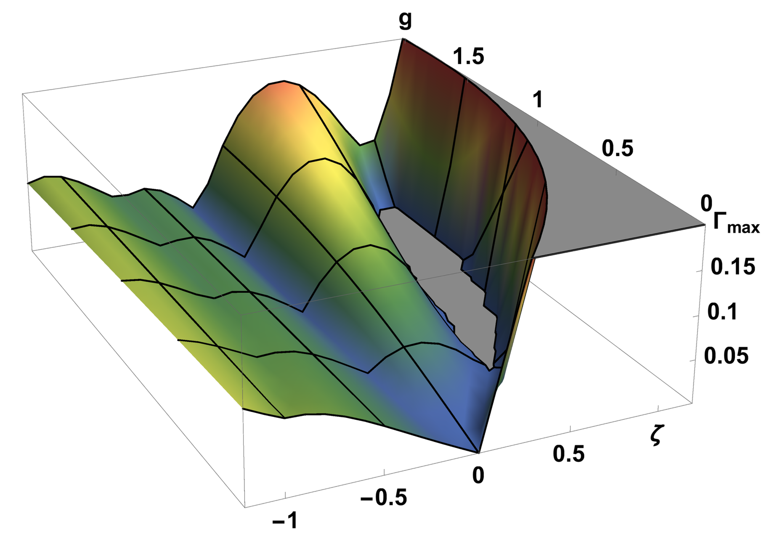
$$F(\tau) = \int_\tau^\infty D(\tau-\sigma)\,\rho(\sigma)\,\mathrm{d}\sigma$$ **— is the quadrupole wake field along the bunch**

$$\mathrm{Z}^\perp(\omega) = i\int_{-\infty}^\infty W(\tau)e^{-i\,\omega\,\tau}\,\mathrm{d}\tau$$ **— is the transverse impedance.**

3. Applications

Without damper when $\zeta \neq 0$ the beam is unstable for all values of $\varkappa$. An example of modeling of coherent growth rates as functions of the head-tail phase for single parabolic bunch in CERN SPS ring is presented to the right. It looks like one should operate an accelerator at $\zeta < 0$ when below the transition energy (and $\zeta > 0$ above the transition) in order to minimize the most unstable growth rate. At the same time an opposite to conventional sign of chromaticity has a hidden advantage: for small values of $|\zeta|$ the only unstable mode is 0-th. This rather general case gives a hope that the use of resistive damper will help to stabilize the beam since 0-th mode is visible well (see [4] for more details).





Left figure shows another example: single Gaussian bunch in FNAL Recycler (resistive wall wake). Now the only growth rate of the most unstable mode as a function of the head-tail phase and gain of resistive damper is plotted. As has been expected for $0 < \zeta < 1$ and $g \sim 1$ there is a "Lake of Stability" where all modes have negative growth rates [4].

[1] A. Burov, Phys. Rev. ST Accel. Beams **12**, 044202, (2009).
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[3] A. Burov, arXiv:1505.07704, (2015).
[4] A. Burov, Phys. Rev. Accel. Beams **19**, 084402, (2016).