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SCHARGEV 1.0 - STRONG SPACE CHARGE VLASOV SOLVER *

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Abstract

The space charge (SC) is known to be one of the major limitations for the collective transverse beam stability. When space charge is strong, i.e. SC tune shift ≫ synchrotron tune Q_s, the problem allows an exact analytical solution. For that practically important case we present a fast and effective Vlasov solver SCHARGEV (Space CHARGE Vlasov) which calculates a complete eigensystem (spatial shapes of modes and frequency spectra) and therefore provides the growth rates and the thresholds of instabilities. SCHARGEV 1.0 includes driving and detuning wake forces, and, any feedback system (damper). In the next version we will include coupled bunch interaction and Landau damping. Numerical examples for FermiLab Recycler and CERN SPS are presented.

INTRODUCTION

SCHARGEV 1.0 is based on SSC theory developed in [1–3]. In this section we will briefly summarize its results for a single bunch with longitudinal distribution function $f(\tau, \mathrm{v})$ where τ is the position along the bunch measured in radians and v is the particle longitudinal velocity. Solutions describing transverse modes for zero-wake case (SSC harmonics $[\nu_k, Y_k(\tau)]$) form an orthonormal basis $\int \rho(\tau)\,Y_l(\tau)\,Y_m(\tau)\,\mathrm{d}\,\tau = \delta_{lm}$ and satisfy

$$\begin{cases} \dfrac{1}{Q_{\mathrm{eff}}(\tau)}\dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\dfrac{\mathrm{d}\,Y(\tau)}{\mathrm{d}\tau}\right) + \nu\,Y(\tau) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\,Y(\tau)\Big|_{\tau=\pm\infty} = 0, \end{cases}$$

where Q_{eff} is the effective space charge tune shift

$$Q_{\mathrm{eff}}(\tau) = Q_{\mathrm{eff}}(0)\,\frac{\rho(\tau)}{\rho(0)},$$

ρ is the normalized line density

$$\rho(\tau) = \int f(\tau,\mathrm{v})\,\mathrm{d}\,\mathrm{v}: \qquad \int \rho(\tau)\,\mathrm{d}\,\tau = 1,$$

and temperature function u^2 is the average square of particle longitudinal velocity

$$u^2(\tau) = \int f(\tau,\mathrm{v})\,\mathrm{v}^2\,\mathrm{d}\,\mathrm{v}\Big/\rho(\tau).$$

The modified dynamic equation including the wake and the damper is

$$\frac{1}{Q_{\mathrm{eff}}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2(\tau)\frac{\mathrm{d}\,\mathcal{Y}(\tau)}{\mathrm{d}\tau}\right) + \omega\,\mathcal{Y}(\tau) = \left[\varkappa\left(\widehat{\mathrm{W}}+\widehat{\mathrm{D}}\right) - i\,g\,e^{i\,\psi}\,\widehat{\mathrm{G}}\right]\mathcal{Y}(\tau)\,.$$

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The operators of wake forces are defined in terms of driving and detuning wakes, respectively

$$\widehat{\mathrm{W}}\,Y(\tau) = \int_\tau^\infty W(\tau-\sigma)\,\rho(\sigma)\,Y(\sigma)\,e^{i\,\zeta(\tau-\sigma)}\,\mathrm{d}\,\sigma,$$

$$\widehat{\mathrm{D}}\,Y(\tau) = Y(\tau)\int_\tau^\infty D(\tau-\sigma)\,\rho(\sigma)\,\mathrm{d}\,\sigma,$$

where the lower limit guarantees the causality, i.e. $\sigma > \tau$. $\zeta = -\xi/\eta$ is the negated ratio of conventional chromaticity, ξ, and a slippage factor, $\eta = \gamma_{\mathrm{t}}^{-2} - \gamma^{-2}$.

$$\varkappa = N_{\mathrm{b}}\,\frac{r_0 R_0}{4\,\pi\,\gamma\,\beta^2\,Q_\beta}$$

where N_{b} is the number of particles per bunch, r_0 is the classical radius of the beam particle, R_0 is the average accelerator ring radius, Q_β as the bare betatron tune, γ is Lorentz factor and β is the ratio of particle velocity to speed of light.

The operator of damper is defined through the pickup $P(\tau)$ and kicker $K(\tau)$ functions

$$\widehat{\mathrm{G}}\,Y(\tau) = K(\tau)\int_{-\infty}^\infty P(\sigma)\,\rho(\sigma)\,Y(\sigma)\,e^{i\,\zeta(\tau-\sigma)}\,\mathrm{d}\,\sigma.$$

g and ψ are the dimensionless gain and the damper's phase. The convention is such that resistive damper defined as $g > 0$ and $\psi = 0$, and, $\psi = \pm\pi/2$ are focusing and defocussing reactive dampers respectively.

Expansion over the zero-wake basis of SSC harmonics

$$\mathcal{Y}_k(\tau) = \sum_{i=0}^\infty \mathbf{C}_i^{(k)} Y_i(\tau)$$

leads to the eigenvalue problem $\mathbf{M}\cdot\mathbf{C}^{(k)} = \omega_k\,\mathbf{C}^{(k)}$ where the matrix $\mathbf{M}$ depends on head-tail phase ζ

$$\mathbf{M}_{lm} = \nu_l\,\delta_{lm} + \varkappa\left[\widehat{\mathrm{W}}_{lm}(\zeta) + \widehat{\mathrm{D}}_{lm}\right] - i\,g\,e^{i\,\psi}\,\widehat{\mathrm{G}}_{lm}(\zeta)$$

with matrix elements being

$$\widehat{\mathrm{W}}_{lm} = \int_{-\infty}^\infty\int_\tau^\infty \mathrm{d}\,\sigma\,\mathrm{d}\,\tau\; W(\tau-\sigma)\,\rho(\tau)\,\rho(\sigma)\,Y_l(\tau)\,Y_m(\sigma)\,e^{i\,\zeta(\tau-\sigma)},$$

$$\widehat{\mathrm{D}}_{lm} = \int_{-\infty}^\infty\int_\tau^\infty \mathrm{d}\,\sigma\,\mathrm{d}\,\tau\,D(\tau-\sigma)\,\rho(\tau)\,\rho(\sigma)\,Y_l(\tau)\,Y_m(\tau),$$

$$\widehat{\mathrm{G}}_{lm} = \mathrm{K}_l(\zeta)\,\mathrm{P}_m^*(\zeta), \quad \text{where}$$

$$\mathrm{P}_k = \int_{-\infty}^\infty \mathrm{d}\,\tau\,P(\tau)\,\rho(\tau)\,Y_k(\tau)\,e^{i\,\zeta\,\tau},$$

$$\mathrm{K}_k = \int_{-\infty}^\infty \mathrm{d}\,\tau\,K(\tau)\,\rho(\tau)\,Y_k(\tau)\,e^{i\,\zeta\,\tau}.$$

For the sake of convenience we will separate real and imaginary parts of eigenvalues and denote them as $\omega_k = \Delta_k + i\,\Gamma_k$.

SCHARGEV 1.0

SCHARGEV has two models of longitudinal phase-space: Gaussian and Hoffman-Pedersen distribution functions

$$f^{(G)}(\tau, v) = \frac{1}{2\pi a b} \exp\left[-\frac{\tau^2}{2a^2} - \frac{v^2}{2b^2}\right],$$
$$f^{(HP)}(\tau, v) = \frac{3}{2\pi a b}\sqrt{1 - \frac{\tau^2}{a^2} - \frac{v^2}{b^2}}\, H\left[1 - \frac{\tau^2}{a^2} - \frac{v^2}{b^2}\right],$$

where $H(\tau)$ is the Heaviside step function. First one represents a thermalized beam (the average square of the particle longitudinal velocity is constant) with Gaussian line density

$$\rho(\tau) = \frac{1}{\sqrt{2\pi}\, a} \exp\left[-\frac{\tau^2}{2a^2}\right], \qquad u^2(\tau) = b^2,$$

while for Hoffman-Pedersen distribution the average square of velocity, as well as line density, is quadratic with τ

$$\rho(\tau) = \frac{3}{4a}\left(1 - \frac{\tau^2}{a^2}\right), \qquad u^2(\tau) = \frac{b^2}{4}\left(1 - \frac{\tau^2}{a^2}\right).$$

Using normalized variables where τ is measured in units of a and ν in units of $Q_s^2/Q_{\rm eff}(0)$, the eigenfunction equations for transverse bunch oscillations are

$$\begin{cases} \dfrac{d}{d\tau}\left[\dfrac{d}{d\tau} Y(\tau)\right] + \nu\, e^{-\tau^2/2}\, Y(\tau) = 0, \\ Y'(\pm\infty) = 0, \end{cases}$$
$$\begin{cases} \dfrac{d}{d\tau}\left[(1-\tau^2)\dfrac{d}{d\tau} Y(\tau)\right] + 4\,\nu\,(1-\tau^2)\, Y(\tau) = 0, \\ Y'(\pm 1) = 0. \end{cases}$$

Both eigenproblems were solved numerically for first 40 modes using MATHEMATICA 10.0. First few eigenvalues and eigenfunctions are listed in Table 1 and presented in Fig. 1.

Table 1: Eigenvalues ν_k for Gaussian and Parabolic Bunches.

k	0	1	2	3	4	5
$\nu_k^{(G)}$	0	1.342	4.325	8.898	15.053	22.787
$\nu_k^{(HP)}$	0	1.156	3.591	7.271	12.191	18.347

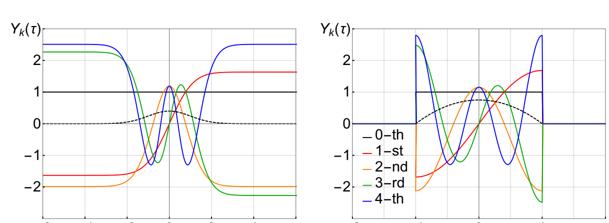

Figure 1: First five SSC harmonics $Y_k(\tau)$ for Gaussian (left) and Hoffman-Pedersen (right) longitudinal distribution functions. Dashed black line shows line density of a bunch.

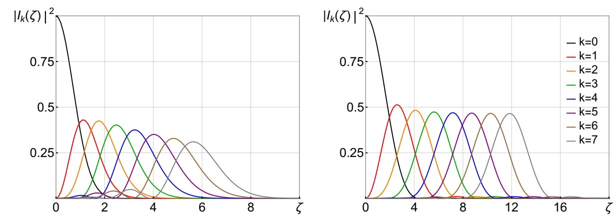

Figure 2: Chromatic factors at strong space charge as functions of head-tail phase for Gaussian (left) and Hoffman-Pedersen (right) longitudinal distribution functions.

Bunch Dipole Moments and Damper

SSC harmonics defines dipole moments characterizing contribution of the specific harmonic to the motion of a total center of mass. As functions of the head-tail phase they are

$$I_k(\zeta) = \int_{-\infty}^{\infty} \rho(\tau)\, Y_k(\tau)\, e^{i\,\zeta\,\tau}\, d\tau,$$

satisfying following parity property

$$I_k^*(\zeta) = I_k(-\zeta) = (-1)^k I_k(\zeta).$$

Corresponding functions were numerically evaluated for $\zeta \in [-50, 50]$ and its absolute squares are presented in Fig. 2.

When damper is *bunch-by-bunch* and *flat* (it sees the center of mass of an individual bunch and applies a dipole kick uniformly along its length) the pickup and kicker functions are $P, K(\tau) = 1$. In this case the damper can be expressed as a matrix of direct product of dipole moments

$$\begin{aligned}\widehat{G}_{lm}(\zeta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau\, d\sigma\, \rho(\tau)\rho(\sigma) Y_l(\tau) Y_m(\sigma)\, e^{i\,\zeta(\tau-\sigma)} \\ &= I_l(\zeta) I_m^*(\zeta) = (-1)^m I_l(\zeta) I_m(\zeta).\end{aligned}$$

Wake Forces

Double integrals in expressions for driving and detuning wake matrix elements can be reduced to single

$$\widehat{W}_{lm}(\zeta) = i\,(-1)^{m+1}\int_{-\infty}^{\infty} Z^{\perp}(\omega-\zeta) I_l(\omega) I_m(\omega)\,\frac{d\omega}{2\pi},$$
$$\widehat{D}_{lm} = \int_{-\infty}^{\infty} F(\tau)\rho(\tau) Y_l(\tau) Y_m(\tau)\, d\tau,$$

where $F(\tau) = \int_{\tau}^{\infty} D(\tau-\sigma)\,\rho(\sigma)\, d\sigma$ is the quadrupole wake field along the bunch and transverse impedance is

$$Z^{\perp}(\omega) = i\int_{-\infty}^{\infty} W(\tau) e^{-i\,\omega\,\tau}\, d\tau.$$

Using these expressions any wake field can be used. SCHARGEV's default library includes matrix elements for model constant and oscillating wakes, and, more realistic resistive wall and broad-band resonator wake fields.

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APPLICATIONS

SCHARGEV 1.0 is equipped with a library of precalculated matrices and solves the eigenvalue problem for matrix with

$$\mathbf{M}_{lm} = \nu_l\, \delta_{lm} + \varkappa \left[\widehat{\mathrm{W}}_{lm}(\zeta) + \widehat{\mathrm{D}}_{lm}\right] - i\, g\, e^{i\,\psi}\widehat{\mathrm{G}}_{lm}(\zeta)\,.$$

Below we will demonstrate how to analyze the beam stability using FermiLab Recycler and CERN SPS impedance models (for simplicity we will neglect the action of detuning wake).

Without damper when $\zeta \neq 0$ the beam is unstable for all values of $\varkappa$. An example of coherent growth rates as functions of the head-tail phase for single parabolic bunch in CERN SPS ring is presented in Fig. 3. Figures suggest the conventional logic: to operate an accelerator at $\zeta < 0$ when below the transition energy (and $\zeta > 0$ above the transition) in order to minimize the most unstable growth rate. At the same time an opposite to conventional sign of chromaticity has a hidden advantage: for small values of $|\zeta|$ the only unstable mode is 0-th. This rather general case gives a hope that the use of resistive damper will help to stabilize the beam since 0-th mode is visible well (see [4] for more details).

Figure 4 shows another example: single Gaussian bunch in FNAL Recycler (resistive wall wake). Now the only growth rate of the most unstable mode as a function of the head-tail phase and gain of resistive damper is plotted. As has been expected for $0 < \zeta < 1$ and $g \sim 1$ there is a "Lake of Stability" where all modes have negative growth rates [4].

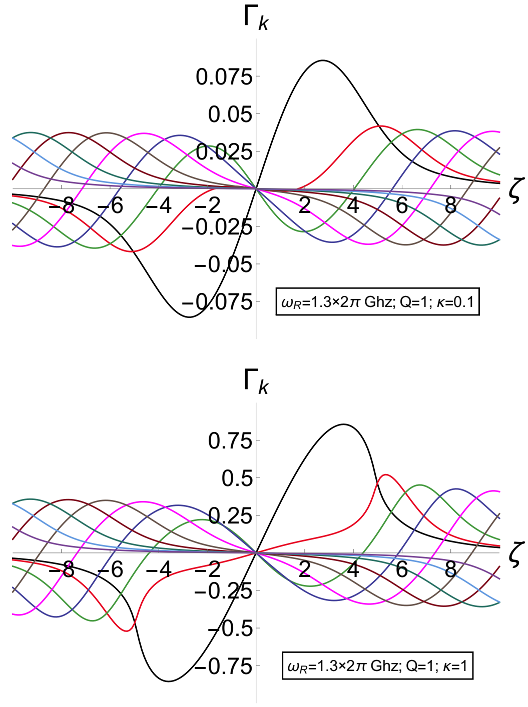

Figure 3: Coherent growth rates for the parabolic bunch with the wake function corresponding to CERN SPS impedance (broad-band resonator $\omega_{\mathrm{R}} = 1.3 \times 2\,\pi$ GHz) as functions of the head-tail phase. Top and bottom figures show different intensities.

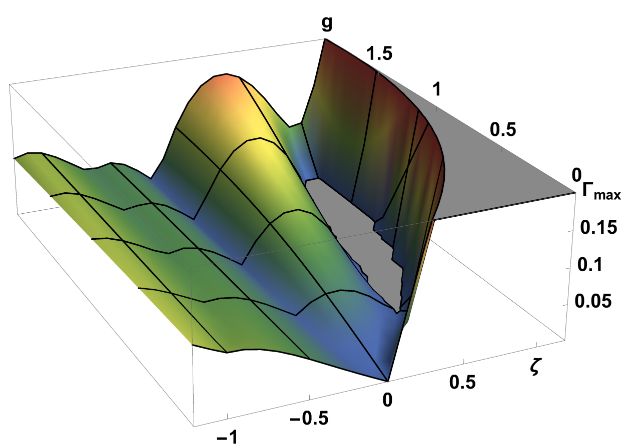

Figure 4: Growth rate of the most unstable mode for the Gaussian bunch with the resistive wall wake as a function of the head-tail phase and gain of resistive flat bunch-by-bunch damper. Gray color shows the "Lake of Stability".

CONCLUSION

The fast and efficient Vlasov solver SCHARGEV 1.0 with two models of longitudinal phase-space has been created. Gaussian distribution represents thermalized bunch, while Hoffman-Pedersen distribution is a good model for well collimated beam. The code includes driving and detuning wake functions and any feedback. The work on couple-bunch wakes and Landau damping is in progress.

As a first result we observed that resistive bunch-by-bunch damper will stabilize the single-bunch instability for $0 < \zeta < 1$. **Note that the chromaticity sign is opposite to conventional**.

In a similar manner SCHARGEV allows to quickly predict growth rates and analyze the bunch spectra for wide range of impedance functions and various parameters of a feedback system.

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