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BEAM DYNAMICS FRAMEWORK INCORPORATING ACCELERATION USED TO DEFINE THE MINIMUM APERTURE OF RF CAVITY FOR FODO-LIKE FOCUSING SCHEME FOR PROTON RADIOTHERAPY LINAC

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Abstract

In this paper, we present a generalised analytical framework for beam dynamics studies and lattice designs, while incorporating longitudinal acceleration of bunches of charged particles. We study a 'FODO-like' scheme, whereby we have an alternating array of focusing and defocusing quadrupoles and study how this differs from a standard FODO lattice due to acceleration. We present optimisation techniques to provide quadrupole parameters, cavity lengths, and required drift lengths under different constraints.

INTRODUCTION

In the recent decades improvements in particle accelerator technology and understanding have allowed a surge in applications to medicine. Two areas to have benefited from such improvements are cancer Radiotherapy and Medical imaging [1, 2]. An important figure of merit of a single RF cell is the shunt impedance, which wants to be maximised. A common method to increase shunt impedance for a given frequency is to reduce the beam aperture. The beam aperture can not be reduced indefinitely as peak surface fields, coupling requirements and most importantly, beam losses, limit the aperture radius. The premise of this paper is to calculate the minimum beam aperture that can be realised with respect to beam losses in a FODO-like scheme factoring in longitudinal acceleration. An accelerating RF cavity map is produced to allow for longitudinal acceleration of protons. Space-charge effects and electromagnetic field effects are ignored. The Twiss parameter mapping matrix is redefined to account for the increase in energy as a proton beam passes through a cavity. The mapping matrix as a function of the betatron phase advance, μ is also redefined to be consistent with increasing momentum. The method implemented minimises the beta function, β, at the cavity entrance and exit in order to maximise beam acceptance transversely.

TRANSVERSE BEAM DYNAMIC RESULTS WITH ACCELERATION

RF Cavity Map

Consider a particle traveling along the z axis and that $p_x << p_z$. If the particle is given a longitudinal kick, p_x is unchanged ad p_z increases by δp_z. If it is assumed the particle Lorentz factor, γ_r, increases linearly in an RF cavity from z = 0 to z = L_{cav} a transverse phase space map for an RF cavity can be shown to take the form of Eq. (1)

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & l_{cav}\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}-\gamma_{r0}} ln\left(\frac{\gamma_{r1}\beta_{r1}+\gamma_{r1}}{\gamma_{r0}\beta_{r0}+\gamma_{r0}}\right) \\ 0 & \frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{1}$$

where β_{r0}/β_{r1} is the normalised longitudinal velocity of a particle before/after the cavity. The determinant of this map is non-unit and therefore it is non symplectic. A bunch of particles in phase space occupying an area, A, will not be a constant of motion along a cavity [3].

The Twiss matrix defines the evolution of the Twiss parameters [4] β, α, γ from some point in a system. The matrix elements are strictly a function of the transfer map between the two points. Considering a system of maps where the only non-symplectic map is a cavity map, the Twiss matrix takes the form of Eq. (2).

The phase advance of a beam element represents the increase in the action angle variable of a particle. Normalising the transverse phase space ellipse using the normalising matrix will produce a phase space circle with the same area. The phase advance can be described as the rotation angle around the phase space circle. A transfer map of an element can, in general, be written as a function of the phase advance. When a cavity map is part of a beam line the total transfer map accumulates an additional term due to acceleration and takes the form shown in Eq. (3).

HALF-FODO CELL

The aim of this paper is to provide the quadrupole magnet strength and length such that the beam size is minimised at the cavity entrance for a FODO-like scheme.

The starting point of this scheme assumes that we at a point where α_x is 0 in the x transverse direction and β_x is at an extremal. We are free to define β_x = maximum. It is convenient to fix the transverse y beam dynamics to be minimum at this exact point: $\alpha_y = 0, \beta_y$ = minimum. Our starting point is therefore some point in a focusing quadrupole of length l_{q1} and k-strength k_1. We can produce the Twiss parameters at any point in a half-FODO cell, such as the cavity entrance, as well as the Twiss parameters at the end of the half-FODO at which point $\alpha_x = \alpha_y = 0$. A schematic of a half-FODO cell is shown in Fig. 1.

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$$\begin{pmatrix}\beta_{x1}\\ \alpha_{x1}\\ \gamma_{x1}\end{pmatrix} = \frac{\gamma_{r1}\beta_{r1}}{\gamma_{r0}\beta_{r0}}\begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2\\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22}\\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2\end{pmatrix}\begin{pmatrix}\beta_{x0}\\ \alpha_{x0}\\ \gamma_{x0}\end{pmatrix} \tag{2}$$

$$\begin{pmatrix}x_1\\ x_1'\end{pmatrix} = \frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}\begin{pmatrix}\sqrt{\frac{\beta_{x1}}{\beta_{x0}}}\,(cos(\mu_x)+\alpha_{x0}sin(\mu_x)) & \sqrt{\beta_{x0}\beta_{x1}}\\ \frac{(\alpha_{x0}-\alpha_{x1})cos(\mu_x)-(1+\alpha_{x0}\alpha_{x1})}{\sqrt{\beta_{x0}\beta_{x1}}} & \sqrt{\frac{\beta_{x0}}{\beta_{x1}}}\,(cos(\mu_x)-\alpha_{x1}sin(\mu_x))\end{pmatrix}\begin{pmatrix}x_0\\ x_0'\end{pmatrix} \tag{3}$$

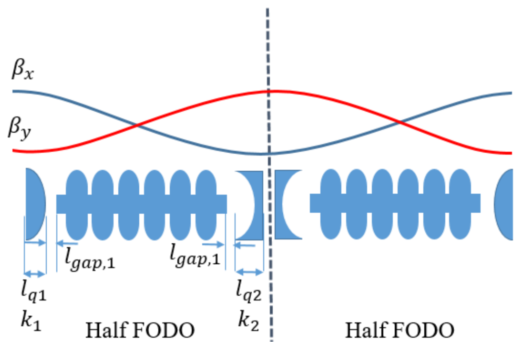

Figure 1: Schematic showing 2 half-FODO cells and the beta functions.

where the centre modified cavity map is a cavity map sandwiched between two drift lengths. The map is similar to Eq. (1) with the 12 element replaced with L_{eff}:

$$L_{eff} = l_{gap}\left(\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}+1\right)+ \\ l_{cav}\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}-\gamma_{r0}}ln\left(\frac{\gamma_{r1}\beta_{r1}+\gamma_{r1}}{\gamma_{r0}\beta_{r0}+\gamma_{r0}}\right) \tag{4}$$

The beta function at the cavity entrance in x is given

$$\beta_{xc0} = R_{11}^2\beta_{x0}+\frac{R_{12}^2}{\beta_{x0}} \tag{5}$$

where R is a transfer map from combining a focusing quadrupole and drift length.

We enforce the beam size in x at the cavity entrance, σ_{x0}, is equal to y beam size, σ_{y1}, at the end of the half-FODO. Requiring the beta function to increase with longitudinal momentum, see Fig. 2.

$$\frac{\gamma_{r1}\beta_{r1}}{\gamma_{r0}\beta_{r0}}\beta_{x0} = \beta_{y1}, \qquad \frac{\gamma_{r1}\beta_{r1}}{\gamma_{r0}\beta_{r0}}\beta_{y0} = \beta_{x1} \tag{6}$$

It is useful to define the ratio of the transverse beam sizes at any point, r, which we require stays constant at the start and end of any half-FODO cell:

$$\frac{\beta_{x0}}{\beta_{y0}} = \frac{\beta_{y1}}{\beta_{x1}} = \frac{\beta_{x2}}{\beta_{y2}} = ... = r \tag{7}$$

The form of the Twiss parameters at the end of a half-FODO cell are found using Eq. (2). Using the results from this and the fact $\det(M_x) = \det(M_y)$ provides useful insight into transfer map element constraints:

$$M_{12} = \pm M_{34}, \qquad M_{21} = \pm M_{43} \tag{8}$$

$$M_{11}M_{22} = M_{33}M_{44} \tag{9}$$

where M represents a (4x4) transfer map for propagating through any odd number of half-FODO cells. We simplify the problem by implementing the semi-thin lens approximation, which expands trigonometric and hyperbolic functions truncating terms of the order $k^n l_{q1}^{n+2}$ and $k_1^n l_{gap}^{n+2}$.

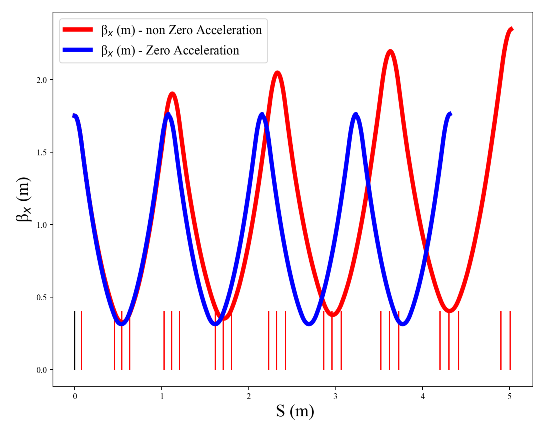

Figure 2: Plot displaying calculated value of β_x along FODO-like beam line comprised of 4 FODO cells for both accelerating and non accelerating scheme.

The constraints lead to the following requirements for a half-FODO cell.

$$l_{q1} = \frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}l_{q2} \tag{10}$$

$$k_1 = \frac{k_2}{\left(\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}\right)^2} \tag{11}$$

$$\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}k_1 l_{q1} = k_2 l_{q2} \tag{12}$$

The above results also hold true if the calculation is carried out to full order. The value of the beta functions at the start of the half-FODO cell can be shown as:

$$\beta_{x0} = \frac{\sqrt{r}}{k_1 l_{q1}}\sqrt{1+\frac{l_{q1}}{L_{eff,1}}} \tag{13}$$

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$$\beta_{y0} = \frac{1}{\sqrt{r}k_1 l_{q1}}\sqrt{1+\frac{l_{q1}}{L_{eff,1}}} \tag{14}$$

In order to find the minimum aperture possible for a given cavity length, a form of the beta function must be found at the start of the cavity in x, and end of the cavity in y. We minimise β_{xc0} (Eq. (5)) by differentiating with respect to k_1 (we can also minimise with respect to the quadrupole length) and keeping l_{q1} and $l_{gap,1}$ as user defined variables.

Substituting and simplifying produces a cubic in k_1 that can be solved analytically [5] to give the lattice parameters that minimise the beam size and keep it constant at any cavity entrance/exit. See Fig. 3.

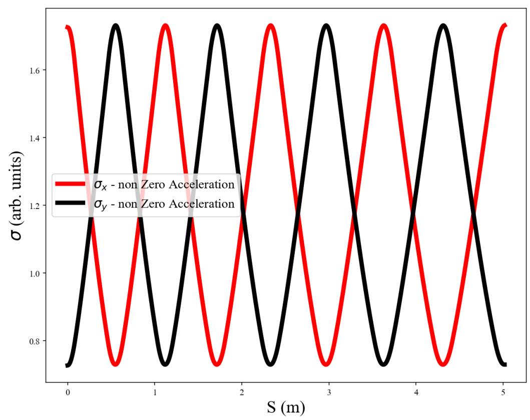

Figure 3: Plot displaying calculated value of σ along FODO-like beam line comprised of 4 FODO cells in accelerating scheme.

BOLTING MULTIPLE HALF-FODO CELLS

Ensuring the constraints found are met for any half-FODO cell we can form propagating equations. We first re-define quadrupole lengths to incorporate a second index that describes if it is the first or second half of a quadrupole magnet

$$l_{q1} = l_{q1,2}, \quad l_{q2} = l_{q2,1}$$

where the second index describes which section of the quadrupole the length describes (first or second).

As each half-FODO cell is made up of 2 half quadrupoles, we can define the

$$k_1 = \frac{k_N}{\Pi_i^N \left(\frac{\gamma_{r(i-1)}\beta_{r(i-1)}}{\gamma_{ri}\beta_{ri}}\right)^2} \tag{15}$$

$$l_{qn,2} = \frac{\gamma_{r(n-1)}\beta_{r(n-1)}}{\gamma_{rn}\beta_{rn}} l_{q(n+1),1} \tag{16}$$

We use the fact $k_2 = k_3$; as they are two sections of the same quadrupole, separated into two half-FODO sections. Equation (16) tells us the relationship between the quadrupole lengths in the n_{th} half-FODO cell.

We now calculate the first quadrupole length in the next half-FODO cell by re-indexing Eqs (13, 14) to describe β_{x1} and β_{y1} as functions of $k_3, k_4, l_{q2,2}, l_{q3,1}, l_{cav,2}, l_{gap,2}$.

A solution is achieved by forcing

$$l_{q2,2} = \frac{l_{q1,2}}{\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}} \quad \text{and} \quad L_{eff,2} = \frac{L_{eff,1}}{\frac{\gamma_{r0}\beta_{r0}}{\gamma_{r1}\beta_{r1}}}$$

This special case solution ensures $l_{qn,1} = l_{qn,2}$ and sets the maximum beam size to occur directly at the midpoint of a quadrupole as is the case in the non-accelerating FODO scheme. The term on the RHS can be solved by equating each term in L_{eff} (relating l_{gap} and l_{cav}) to give $l_{cav,2}, l_{gap,2}$. We also required that the total energy meets the design requirements. As cavity lengths are functions of previous cavity lengths, we can sweep the first cavity length until the sum of cavity lengths provides the correct energy gain. When incorporating longitudinal acceleration, the value of lattice parameters change across a FODO beam line, which can be seen in Fig. 4.

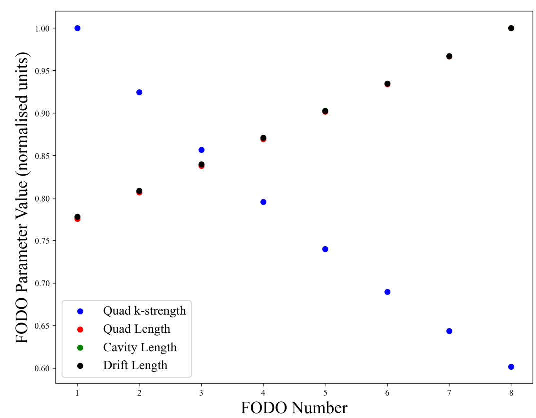

Figure 4: Plot displaying how the lattice parameters vary along a FODO beam line of multiple FODO cells.

CONCLUSION

In this paper, we present a generalised analytical framework for transverse beam dynamics studies and lattice designs incorporating longitudinal acceleration. A 'FODO-like' focusing scheme is studied and the quadrupole lengths and k-strengths are calculated such that the beam size is minimum at a cavity entrance. The Twiss beta function must increase with longitudinal momentum in order to keep the minimum beam size constant for any given cavity entrance. We solve for a specific case by defining the first quadrupole length, drift length and cavity gradient. By calculating the k_1 that minimises the beam, we can use a set of iterative equations that define all FODO beam line parameters that also produce the required energy gain.

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