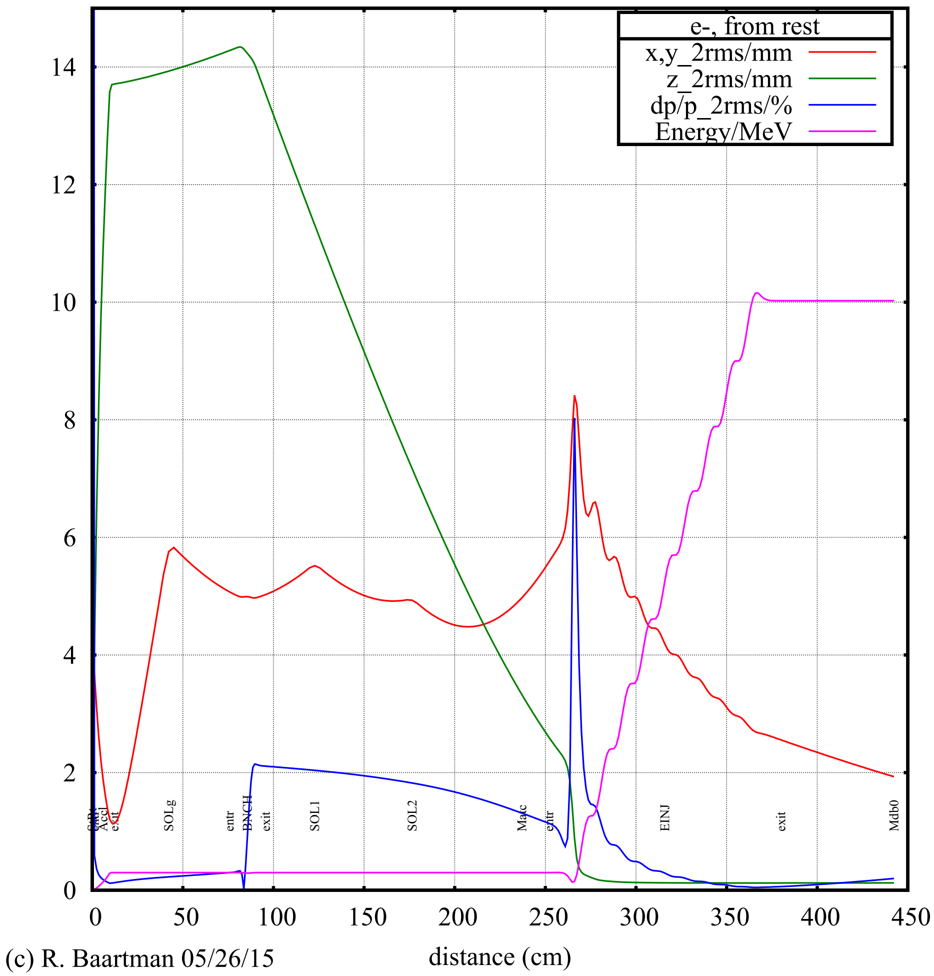
Fast Envelope Tracking for Space Charge Dominated Injectors

Rick Baartman, TRIUMF

September 30, 2016

Introduction



E.g.: Electron gun, solenoids, buncher, linac; $15\,$pC bunch charge.

Want **operator** to:

- Find peak energy gain phase.
- Solenoid match.
- Vary with bunch charge.
- Buncher phase.
- For different cavity excitation.
- Etc.

Outline

1. Sacherer Envelope Theory Review
2. General Hamiltonian-Based Formalism
3. Space Charge
4. Application to Linacs: Hamiltonian 1
5. Hamiltonian 2
6. Some TRIUMF Example Cases
7. GUI

Frank Sacherer:

Instead of multiparticle simulations, we look at the "envelope", i.e. treat the beam statistically

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

96

CERN/SI/Int. DL/70-12
18.11.1970

RMS ENVELOPE EQUATIONS WITH SPACE CHARGE

Frank J. Sacherer

ABSTRACT

The envelope equations for a continuous beam with circular symmetry but otherwise arbitrary charge distribution have been derived by Lapostolle and Gluckstern. Their results are extended in this report to continuous beams with elliptical symmetry and to bunched beams with ellipsoidal form.

Statistical Approach to Beam Dynamics

If there is a distribution of particles, one would like to calculate the final distribution from the initial. The behaviour of the beam centroid

$$\langle \mathbf{X} \rangle = \sum_{i=1}^{N} \mathbf{X}/N \tag{1}$$

(where N is the number of particles, and $\mathbf{X}$ is the column vector $(x, P_x, y, P_y, z, P_z)^T$ as in eqn. 6) is determined by the same transfer matrix $\mathbf{M}$ as for an individual particle. This is the equation of 'first moments'. At the next level, one would like to calculate the evolution of the beam widths, or, 'second moments' given by

$$\boldsymbol{\sigma} \equiv \frac{1}{N} \sum_{i=1}^{N} \mathbf{X}\mathbf{X}^T \tag{2}$$

For example, $\sigma_{11} = \langle x^2 \rangle$, $\sigma_{12} = \langle x P_x \rangle$, $\sigma_{13} = \langle xy \rangle$, ... For a distribution of particles so dense that we do not see graininess on any scale of our diagnostics, the sums go over into integrals. For example,

$$\sigma_{12} = \int\int\int\int\int\int x P_x \, f(x, P_x, y, P_y, z, P_z) \; dx \, dP_x \, dy \, dP_y \, dz \, dP_z,$$

where f is the distribution in phase space, normalized so that its integral over all 6 phase space dimensions is 1.

Here, s is the independent variable, $z = \beta c \Delta t$, $P_z = (\beta c)^{-1} \Delta E$.

By direct substitution into the definition of $\boldsymbol{\sigma}$, we find

$$\boldsymbol{\sigma}_{\rm f} = \mathbf{M}\boldsymbol{\sigma}_{\rm i}\mathbf{M}^T \tag{3}$$

As well, recalling the infinitesimal transfer matrix $\mathbf{F}$ where $\mathbf{X}' = \mathbf{F}\mathbf{X}$ and the transfer matrix of an infinitesimal length ds is $\mathbf{M} = \mathbf{I} + \mathbf{F}ds$, we find directly

$$\boldsymbol{\sigma}' = \mathbf{F}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{F}^T. \tag{4}$$

This is the **envelope equation**. For the full 6D case, it represents 21 equations. (Because σ is symmetric.)

What is $\mathbb{F}$? Infinitesimal Transfer Matrix

The general Hamiltonian can be Taylor-expanded by orders in the 6 dependent variables[1],

$$H(x_1, x_2, x_3, x_4, x_5, x_6; s) = \sum_i \left.\frac{\partial H}{\partial x_i}\right|_0 x_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 H}{\partial x_i \partial x_j}\right|_0 x_i x_j + \ldots \tag{5}$$

The subscript 0 means that the derivatives are evaluated on the reference trajectory $\forall i,\ x_i = 0$. (Keep in mind though that these partial derivatives in general are functions of the independent variable t or s.)

Terms of first order are eliminated by transforming to a coordinate system measured with respect to the reference trajectory. The remaining terms are second order and higher, and for linear motion, we simply truncate at the second order.

[1] In this shorthand, $x_1 = x, x_2 = P_x, x_3 = y, \ldots$

Then the Hamiltonian looks like $H = Ax^2 + BxP_x + Cxy + \ldots + UP_z^2$: there are 21 independent terms. $A = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}$, and so on; all derivatives are evaluated on the reference trajectory, and may be a function of the independent variable. We know the equations of motion from the Hamiltonian to be: $x' = \partial H/\partial P_x$, $P_x' = -\partial H/\partial x$, etc., where primes denote derivatives w.r.t. the independent variable. Therefore the equations of motion:

$$\begin{pmatrix} x' \\ P_x' \\ y' \\ P_y' \\ z' \\ P_z' \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial P_x \partial x} & \frac{\partial^2 H}{\partial P_x^2} & \frac{\partial^2 H}{\partial P_x \partial y} & \frac{\partial^2 H}{\partial P_x \partial P_y} & \frac{\partial^2 H}{\partial P_x \partial z} & \frac{\partial^2 H}{\partial P_x \partial P_z} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial P_x} & -\frac{\partial^2 H}{\partial x \partial y} & -\frac{\partial^2 H}{\partial x \partial P_y} & -\frac{\partial^2 H}{\partial x \partial z} & -\frac{\partial^2 H}{\partial x \partial P_z} \\ \frac{\partial^2 H}{\partial P_y \partial x} & \frac{\partial^2 H}{\partial P_y \partial P_x} & \frac{\partial^2 H}{\partial P_y \partial y} & \frac{\partial^2 H}{\partial P_y^2} & \frac{\partial^2 H}{\partial P_y \partial z} & \frac{\partial^2 H}{\partial P_y \partial P_z} \\ -\frac{\partial^2 H}{\partial y \partial x} & -\frac{\partial^2 H}{\partial y \partial P_x} & -\frac{\partial^2 H}{\partial y^2} & -\frac{\partial^2 H}{\partial y \partial P_y} & -\frac{\partial^2 H}{\partial y \partial z} & -\frac{\partial^2 H}{\partial y \partial P_z} \\ \frac{\partial^2 H}{\partial P_z \partial x} & \frac{\partial^2 H}{\partial P_z \partial P_x} & \frac{\partial^2 H}{\partial P_z \partial y} & \frac{\partial^2 H}{\partial P_z \partial P_y} & \frac{\partial^2 H}{\partial P_z \partial z} & \frac{\partial^2 H}{\partial P_z^2} \\ -\frac{\partial^2 H}{\partial z \partial x} & -\frac{\partial^2 H}{\partial z \partial P_x} & -\frac{\partial^2 H}{\partial z \partial y} & -\frac{\partial^2 H}{\partial z \partial P_y} & -\frac{\partial^2 H}{\partial z^2} & -\frac{\partial^2 H}{\partial z \partial P_z} \end{pmatrix} \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{pmatrix} \qquad (6)$$

or,

$$\mathbf{X}' = \mathbf{F}\mathbf{X},$$

where $\mathbf{F}$ is called the ‘infinitesimal transfer matrix’. Of the 36 elements of $\mathbf{F}$ there are only 21 independent ones. Easily integrated if $\mathbf{F}$=constant.

Example: Quadrupole

A particular case is where the beamline consists only of elements that keep all 3 degrees of freedom independent of each other, and there is only a focusing force $K(s)$ that varies with s. In other words, the Hamiltonian is 7,

$$H = \frac{P_x^2}{2} + K(s)\frac{x^2}{2} + \frac{P_y^2}{2} - K(s)\frac{y^2}{2} + \frac{P_z^2}{2\gamma^2} \tag{7}$$

so

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

Quadrupole (Hard-edge)

Since infinitesimal matrix is constant, can integrate directly (in Mathematica using MatrixExp):

$$\textbf{MatrixForm}\left[\textbf{ExpToTrig}\left[\textbf{MatrixExp}\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ -K^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & K^2 & 0 \end{pmatrix} L\right]\right]\right]$$

$$\begin{pmatrix} \text{Cos}[K L] & \frac{\text{Sin}[K L]}{K} & 0 & 0 \\ -K\,\text{Sin}[K L] & \text{Cos}[K L] & 0 & 0 \\ 0 & 0 & \text{Cosh}[K L] & \frac{\text{Sinh}[K L]}{K} \\ 0 & 0 & K\,\text{Sinh}[K L] & \text{Cosh}[K L] \end{pmatrix}$$

Example: Solenoid

$K(s) = \frac{B(s)}{2B\rho}$ is often as in `TRANSPORT` approximated as constant (hard-edge). **Not** a good approximation for short solenoids as in electron injectors.

If constant,

$$\boldsymbol{M} = \textbf{Simplify}\left[\textbf{ExpToTrig}\left[\textbf{MatrixExp}\left[\begin{pmatrix} 0 & 1 & K & 0 \\ -K^2 & 0 & 0 & K \\ -K & 0 & 0 & 1 \\ 0 & -K & -K^2 & 0 \end{pmatrix} L\right]\right]\right]; \textbf{MatrixForm}[M]$$

$$\begin{pmatrix} \text{Cos}[K L]^2 & \frac{\text{Sin}[2 K L]}{2K} & \text{Cos}[K L]\text{Sin}[K L] & \frac{\text{Sin}[K L]^2}{K} \\ -\frac{1}{2}K\text{Sin}[2 K L] & \text{Cos}[K L]^2 & -K\text{Sin}[K L]^2 & \text{Cos}[K L]\text{Sin}[K L] \\ -\frac{1}{2}\text{Sin}[2 K L] & -\frac{\text{Sin}[K L]^2}{K} & \text{Cos}[K L]^2 & \frac{\text{Sin}[2 K L]}{2K} \\ K\text{Sin}[K L]^2 & -\frac{1}{2}\text{Sin}[2 K L] & -\frac{1}{2}K\text{Sin}[2 K L] & \text{Cos}[K L]^2 \end{pmatrix}$$

Apply Rotation

$$\text{Rot[A\_]} = \begin{pmatrix} \text{Cos}[A] & 0 & \text{Sin}[A] & 0 \\ 0 & \text{Cos}[A] & 0 & \text{Sin}[A] \\ -\text{Sin}[A] & 0 & \text{Cos}[A] & 0 \\ 0 & -\text{Sin}[A] & 0 & \text{Cos}[A] \end{pmatrix};$$

$$\text{MatrixForm[Simplify[Rot}[-K\,L].M]]$$

$$\begin{pmatrix} \text{Cos}[K\,L] & \frac{\text{Sin}[K\,L]}{K} & 0 & 0 \\ -K\,\text{Sin}[K\,L] & \text{Cos}[K\,L] & 0 & 0 \\ 0 & 0 & \text{Cos}[K\,L] & \frac{\text{Sin}[K\,L]}{K} \\ 0 & 0 & -K\,\text{Sin}[K\,L] & \text{Cos}[K\,L] \end{pmatrix}$$

But what about if $\mathbf{F} = \mathbf{F}(s)$?

Soft-edges?

Space Charge?

Space Charge part of $\mathbf{F}$

The beam is in bunches rather than continuous, so we need the electric field of an ellipsoidal distribution of charge. It turns out, surprisingly (Sacherer, 1971), that the RMS linear part of the space charge self-field depends mainly on the RMS size of the distribution and only very weakly on its exact form. To within a few percent, the RMS linear part of space charge is the same as that for a uniformly populated ellipsoid. The space charge infinitesimal transfer matrix is

$$\mathbf{F}_{\rm sc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ K_{x\rm sc} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{y\rm sc} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{z\rm sc} & 0 \end{pmatrix} \qquad (9)$$

where

$$K_{x\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{a^3}\,g\left(\frac{b^2}{a^2},\frac{c^2}{a^2}\right) \tag{10}$$

$$K_{y\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{b^3}\,g\left(\frac{c^2}{b^2},\frac{a^2}{b^2}\right) \tag{11}$$

$$K_{z\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{c^3}\,g\left(\frac{a^2}{c^2},\frac{b^2}{c^2}\right) \tag{12}$$

where Q is the bunch charge, the ellipsoid semi-axes in the x, y, z directions are a, b, c, and the function g is

$$g(u,v) = \frac{3}{2}\int_0^\infty (1+s)^{-3/2}(u+s)^{-1/2}(v+s)^{-1/2}ds \tag{13}$$

This is from the family of Carlson elliptic integrals.

Arbitrary bunch distributions, orientations

For arbitrary distributions of the type $f(x,y,z) = f\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$, replace a, b, c with the RMS values according to the values they have for the uniform case, namely, $a^2 = 5\sigma_{11}$, $b^2 = 5\sigma_{33}$. Because of relativity, c^2 is a special case: $c^2 = 5\gamma^2\sigma_{55}$.

Notice the recursiveness.

For arbitrary orientations, have to apply a rotation matrix to F, thus making also $F_{23}, F_{25}, F_{41}, F_{45}, F_{61}, F_{63}$ also non-zero.

For further reading, again refer to Sacherer (1971), but also de Jong (1983).

Elaborated for the case with space charge, DC (unbunched), uncoupled, it becomes the (better-known) Kapchinsky-Vladimirsky eqns.

What about: TRANSPORT, TRACE3D, ...?

If all elements are integrable then the transfer matrices $\mathbf{M}$ are known, and they are simply multiplied together to find the matrix of the whole beamline or synchrotron, and the final beam is found from the initial as in 3. This is the traditional approach, e.g. TRANSPORT.

To incorporate space charge, elements were subdivided and appropriate thin defocus lenses inserted.

In TRACE3D, there are space charge **impulses** applied in the approximation of long bunches.

7.0 SPACE-CHARGE IMPULSES

Approximate expressions for the electric field components that are due to a uniformly charged ellipsoid, as given by Lapostolle[13] are as follows:

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c\gamma^2}\frac{(1-f)}{r_x(r_x+r_y)r_z}x \quad ,$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c\gamma^2}\frac{(1-f)}{r_y(r_x+r_y)r_z}y \quad ,$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c}\frac{f}{r_x r_y r_z}z \quad ,$$

where r_x, r_y, and r_z are the semiaxes of the ellipsoid, I is the average electrical current (assuming that a bunch occurs in every period of the RF), λ is the free-space wavelength of the RF, c is the velocity of light, and ε_0 is the permittivity of free space. The form factor f is a function of $p \equiv \gamma r_z / \sqrt{r_x r_y}$. Values for f are given in Table III for specific values of p and $1/p$.

TABLE III: Space-Charge Form Factor

---

13 P. M. Lapostolle, CERN report AR/Int. SG/65-15, Geneva, Switzerland (July 1965).

TRANSOPTR

These techniques are approximate and non-adaptive: Why not use the equations of motion directly? There are only 21 of them. In TRANSOPTR, 4 is solved with a Runge Kutta integrator. This allows not only space charge, but any general case with no closed-form solution to eom's, e.g. varying axial fields, linacs, short-soft-edge quads,...

Original version written by Mark deJong, Ed Heighway at Chalk River, Canada.

2666 IEEE Transactions on Nuclear Science, Vol. NS-30, No. 4, August 1983

A FIRST ORDER SPACE CHARGE OPTION FOR TRANSOPTR

M.S. de Jong and E.A. Heighway

Atomic Energy of Canada Limited, Research Company
Chalk River Nuclear Laboratories
Chalk River, Ontario, Canada K0J 1J0

Successfully applied to:

- beamlines, achromatic fitting, space charge
- complex transport problems such as einzel lenses, soft-landing, into solenoid, cyclotron inflectors
- synchrotrons: finding β-functions with space charge
- linacs...

As well, it has optimization routines; simplex method, simulated annealing.

Application to: Standing Wave Linear Accelerator

Hamiltonian

With the distance along the reference trajectory s as the independent variable, the Hamiltonian is

$$
\begin{aligned}
& H(x, P_x, y, P_y, t, E; s) = \qquad (14) \\
= \quad & -qA_s - \sqrt{\left(\frac{E - q\Phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2}
\end{aligned}
$$

Potentials

The case of RF axially-symmetric electric field can be handled entirely with no electric potential ($\Phi = 0$), and time-varying vector potential. This has been presented a number of times in the past (e.g. E.E. Chambers;1968), but we are interested in the following more experimentally-useful case: The electric field along the axis $\mathcal{E}(s)$ has been measured and is therefore known, and the geometry is exactly axially symmetric.

Rob Ryne(1991) has treated this case, and we use his vector potential $\vec{A}(x, y, s, t)$ directly.

$$
\begin{aligned}
A_x &= \frac{\mathcal{E}'(s)}{2}\frac{\sin(\omega t+\theta)}{\omega}\,x && (15)\\
A_y &= \frac{\mathcal{E}'(s)}{2}\frac{\sin(\omega t+\theta)}{\omega}\,y && (16)\\
A_s &= \left(-\mathcal{E}(s)+\frac{x^2+y^2}{4}\left[\mathcal{E}''(s)+\frac{\omega^2}{c^2}\mathcal{E}(s)\right]\right)\frac{\sin(\omega t+\theta)}{\omega} && (17)
\end{aligned}
$$

This is Coulomb/Lorenz gauge ($\Phi = 0$), satisfies Maxwell equations to second order in transverse coordinates, gives correct on-axis $\vec{\mathcal{E}} = -\partial\vec{A}/\partial t = \mathcal{E}\cos(\omega t+\theta)$.

A Word About Coordinates 5 and 6

SLAC-91 (Karl Brown) mentions “At any position in the system... ”. This means that time t is **NOT** the independent variable. Then goes on: “...particle represented by a vector”:

$$(x, \theta, y, \phi, l, \delta)$$

(where l is trajectory length and $\delta \equiv \Delta P/P$).

This is **wrong**: The canonical pair are $(t - t_0, E - E_0)$ or $(\Delta t, \Delta E)$, **not** $(l, \Delta P/P)$.

The reason it works usually is by applying a trick: If we scale by βc, we can make them match (sort of), since $\beta c \Delta t = z$, $\Delta E/(\beta c) = \Delta P$, but **ONLY TRUE OF STATIC MAGNETIC ELEMENTS**: Electric potential $\Phi = 0,\ \vec{A} \neq \vec{A}(t)$.

Mathematical Formulation of TRANSPORT(*)

The following of a charged particle through a system of magnetic lenses may be reduced to a process of matrix multiplication. At any specified position in the system an arbitrary charged particle is represented by a vector (single column matrix), X, whose components are the positions, angles, and momentum of the particle with respect to a specified reference trajectory.

$$\text{i. e.} \quad X = \begin{bmatrix} x \\ \theta \\ y \\ \varphi \\ \ell \\ \delta \end{bmatrix}$$

Definitions:

x = the radial displacement of the arbitrary ray with respect to the assumed central trajectory.

θ = the angle this ray makes in the radial plane with respect to the assumed central trajectory.

y = the transverse displacement of the ray with respect to the assumed central trajectory.

φ = the transverse angle of the ray with respect to the assumed central trajectory.

(*) For a more complete description of the mathematical basis of TRANSPORT, refer to SLAC Report 75, the Appendix of this report and to other References listed at the end of this report.

ℓ = the path length difference between the arbitrary ray and the central trajectory

δ = $\Delta P/P$ is the fractional momentum deviation of the ray from the assumed central trajectory.

The magnetic lens is represented by a square matrix, R, which describes the action of the magnet on the particle coordinates. Thus the passage of a charged particle through the system may be represented by the equation:

$$X|1| = R\,X|0| \qquad (1)$$

where $X|0|$ is the initial coordinate vector and $X|1|$ is the final coordinate vector of the particle under consideration; R is the transformation matrix for all such particles traversing the system (one particle differing from another only by its initial coordinate vector $X|0|$).

The traversing of several magnets and interspersing drift spaces is described by the same basic equation but with R now being the product matrix R = R(n)...R(3)R(2)R(1) of the individual matrices of the system elements. The following of a charged particle via TRANSPORT through a system of magnets is thus analogous to tracing rays through a system of optical lenses except that TRANSPORT is a matrix calculation which truncates the problem to either first or second-order in a Taylor's expansion about a central trajectory. For studying beam optics to greater precision than a second-order TRANSPORT calculation permits,

But: Don't know t_0 and E_0

A priori, we do not know the reference particle's energy and time coordinates. We need these in order to expand about them. (See eqn. 5.) They can be found from the equations of motion for $x = y = P_x = P_y = 0$:

$$\frac{dE_0}{ds} = \frac{\partial H}{\partial t} = q\mathcal{E}\cos(\omega t_0 + \theta) \tag{18}$$

$$\frac{dt_0}{ds} = -\frac{\partial H}{\partial E} = \frac{E_0}{P_0} = \frac{1}{\beta_0 c} \tag{19}$$

These 2 are added to the 21 mentioned previously; 23 solved together.

(From here on, I drop the 0 subscript: β and γ are implicitly assumed to be the relativistic parameters of the reference particle.)

These give the functions $E_0(s)$ and $t_0(s)$ about which t and E are expanded: $E = E_0 + \Delta E$, $t = t_0 + \Delta t$. So we transform the canonical variables t and $-E$ to $(\Delta t, -\Delta E)$, using as generating function

$$G = -\left(t - \int \frac{ds}{\beta(s)c}\right)(\Delta E + E_0) \tag{20}$$

(Check: $\frac{\partial G}{\partial t} = -E$, $\frac{\partial G}{\partial(-\Delta E)} = \Delta t$.) The Hamiltonian gets the added terms

$$\frac{\partial G}{\partial s} = \frac{\Delta E + E_0(s)}{\beta(s)c} - \Delta t E_0'(s).$$

Then expanding the square root, we get:

$$H(x, P_x, y, P_y, \Delta t, \Delta E; s) = \left(\frac{E_0}{\beta c} - P_0\right) - qA_s - \Delta t E_0'(s) + \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} + \frac{(P_x - qA_x)^2 + (P_y - qA_y)^2}{2P} \tag{21}$$

In expanding $P_x - qA_x$, $P_y - qA_y$, the time dependence disappears because it is higher order:

$$(P_x - qA_x)^2 = P_x^2 - q\mathcal{E}'\frac{\sin(\omega t_0 + \theta)}{\omega} xP_x + \left(\frac{q\mathcal{E}'}{2}\frac{\sin(\omega t_0 + \theta)}{\omega}\right)^2 x^2, \tag{22}$$

and similarly for y. The term linear in Δt in the expansion of A_s about t_0 cancels the $-\Delta t E_0'(s)$ term, as it should but there is a remaining term quadratic in Δt, the bunching effect. This leaves

$$-qA_s - \Delta t E_0'(s) = q\mathcal{E}\frac{\sin(\omega t_0+\theta)}{\omega}\left(1-\frac{\omega^2(\Delta t)^2}{2}\right) - \frac{r^2 q}{4}\left(\mathcal{E}'' + \frac{\omega^2}{c^2}\mathcal{E}\right)\frac{\sin(\omega t_0+\theta)}{\omega} \tag{23}$$

Notice the first term here and the first term in eqn. 21 depend only on the independent variable and not on the 6 dependent ones. Thus these do not affect the equations of motion and we ignore them. We have:

$$\begin{aligned} H(x,P_x,y,P_y,\Delta t,\Delta E;s) \quad = \quad & -\frac{q\mathcal{E}}{2}\omega^2 T(\Delta t)^2 + \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} - \frac{r^2 q}{4}\left(\mathcal{E}'' + \frac{\omega^2}{c^2}\mathcal{E}\right)T \\ & + \frac{P_x^2}{2P} - q\mathcal{E}'T\,\frac{xP_x}{2P} + \left(\frac{q\mathcal{E}'}{2}T\right)^2\frac{x^2}{2P} \\ & + \frac{P_y^2}{2P} - q\mathcal{E}'T\,\frac{yP_y}{2P} + \left(\frac{q\mathcal{E}'}{2}T\right)^2\frac{y^2}{2P} \end{aligned} \tag{24}$$

We defined here $T(s) = \sin[\omega t_0(s)+\theta]/\omega$ to clean up the notation a bit.

Finally, we wish to transform from $(\Delta t, -\Delta E)$ to $(z, P_z) = (-\beta c\Delta t, \Delta E/(\beta c))$. (The reason for the sign change is as follows: an early arrival implies $\Delta t < 0$, but this means the particle is **ahead** so $z > 0$.) The generating function is

$$G = -\beta c\Delta t P_z \tag{25}$$

(Check: $\frac{\partial G}{\partial \Delta t} = -\Delta E$, $\frac{\partial G}{\partial (P_z)} = z$.) The term to be added to the Hamiltonian is

$$\frac{\partial G}{\partial s} = \frac{\beta'}{\beta}\, zP_z = \frac{\gamma'}{\beta^2\gamma^3}\, zP_z = \frac{q\mathcal{E}C}{\beta c P\gamma^2}\, zP_z,$$

where $C \equiv \cos(\omega t_0 + \theta)$.

$$\begin{aligned} H(x, P_x, y, P_y, z, P_z; s) \quad = \quad & \frac{P_x^2}{2P} - q\mathcal{E}'T\,\frac{xP_x}{2P} + \left[\frac{1}{P}\left(\frac{q\mathcal{E}'T}{2}\right)^2 - \frac{T}{2}\left(q\mathcal{E}'' + \frac{\omega^2}{c^2}q\mathcal{E}\right)\right]\frac{x^2}{2} + \\ & \frac{P_y^2}{2P} - q\mathcal{E}'T\,\frac{yP_y}{2P} + \left[\frac{1}{P}\left(\frac{q\mathcal{E}'T}{2}\right)^2 - \frac{T}{2}\left(q\mathcal{E}'' + \frac{\omega^2}{c^2}q\mathcal{E}\right)\right]\frac{y^2}{2} + \\ & \frac{P_z^2}{2\gamma^2 P} + \frac{2q\mathcal{E}C}{\beta c}\,\frac{zP_z}{2\gamma^2 P} - \frac{q\mathcal{E}}{\beta^2 c^2}\omega^2 T\,\frac{z^2}{2} \end{aligned} \tag{26}$$

Hamiltonian 2

Ryne(1991) has a transformation that gets rid of the second derivative of the on-axis elecric field. It's complicated. At the same time he transforms away the adiabatic damping; it's a neat and didactic trick but not strictly necessary for computational purposes. It is simple to just use $P_{x,y,z}$ directly and then just rescale by final P at the end.

But there's an easy way to get rid of the second derivative: it turns out that the vector potential can be simplified if we use a different Gauge.

I propose the following function

$$\Psi(x, y, s, t) = -\frac{\mathcal{E}'}{2}\frac{\sin(\omega t + \theta)}{\omega}\frac{x^2 + y^2}{2} \tag{27}$$

Add the gradient of this function to the previous vector potential (15,16,17). This zeroes both A_x and A_y, leaving

$$A_x = 0,\ A_y = 0,\ A_s = -\mathcal{E}(s)\left(1 - \frac{\omega^2}{c^2}\frac{x^2 + y^2}{4}\right)\frac{\sin(\omega t + \theta)}{\omega} \tag{28}$$

This is considerably simpler, but now there is a scalar potential:

$$\Phi = -\frac{\partial \Psi}{\partial t} = \mathcal{E}' \cos(\omega t + \theta) \frac{x^2 + y^2}{4} \tag{29}$$

Now if we expand the Hamiltonian, we get a different result:

$$\begin{aligned} H(x, P_x, y, P_y, z, P_z; s) \quad = \quad & \frac{P_x^2}{2P} + \frac{P_y^2}{2P} + \frac{q}{2\beta c}\left(\mathcal{E}'C - \mathcal{E}S\frac{\omega\beta}{c}\right)\frac{r^2}{2} + \\ & \frac{P_z^2}{2\gamma^2 P} + \frac{q\mathcal{E}C}{\beta c}\frac{zP_z}{\gamma^2 P} - \frac{q\mathcal{E}\omega S}{\beta^2 c^2}\frac{z^2}{2} \end{aligned} \tag{30}$$

$(C \equiv \cos(\omega t_0(s) + \theta)$, $S \equiv \sin(\omega t_0(s) + \theta))$

This is not only much simpler (P_x and P_y have their usual definitions, no transverse cross terms, no $\mathcal{E}''$), but has nice intuitive explanations for the individual terms. (1) The factor in parentheses represents usual the focal power of an RF gap, e.g. a buncher. (2) Taking the limit as $\omega \to 0$ reproduces precisely the Hamiltonian of the DC accelerator. Note that in that case, $\mathcal{E}' = -\phi''$.

Infinitesimal Transfer Matrix $\mathbf{F}$

Now that the Hamiltonian for linear motion (eqn. 30) has been obtained, it is a simple matter to find the infinitesimal transfer matrix F. Writing the equations of motion ($x' = \partial H/\partial P_x$, $P_x' = -\partial H/\partial x$, etc.), the following F-matrix is found for the axially symmetric linear accelerator:

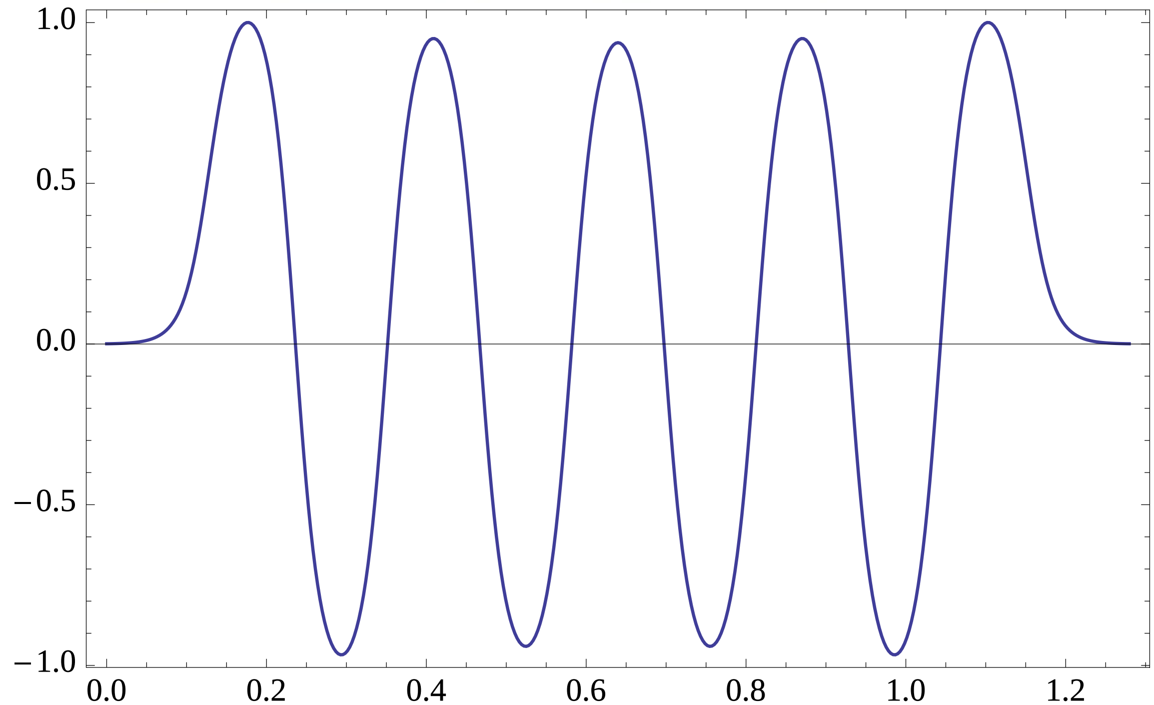
$$F = \begin{pmatrix} 0 & \frac{1}{P} & 0 & 0 & 0 & 0 \\ \mathcal{A}(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{P} & 0 & 0 \\ 0 & 0 & \mathcal{A}(s) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta'}{\beta} & \frac{1}{\gamma^2 P} \\ 0 & 0 & 0 & 0 & \mathcal{B}(s) & -\frac{\beta'}{\beta} \end{pmatrix}. \tag{31}$$

where we have defined:

$$\mathcal{A}(s) = \frac{-q}{2\beta c}\left(\mathcal{E}'C - \mathcal{E}S\frac{\omega\beta}{c}\right),\ \mathcal{B}(s) = \frac{q\mathcal{E}\omega S}{\beta^2 c^2}. \tag{32}$$

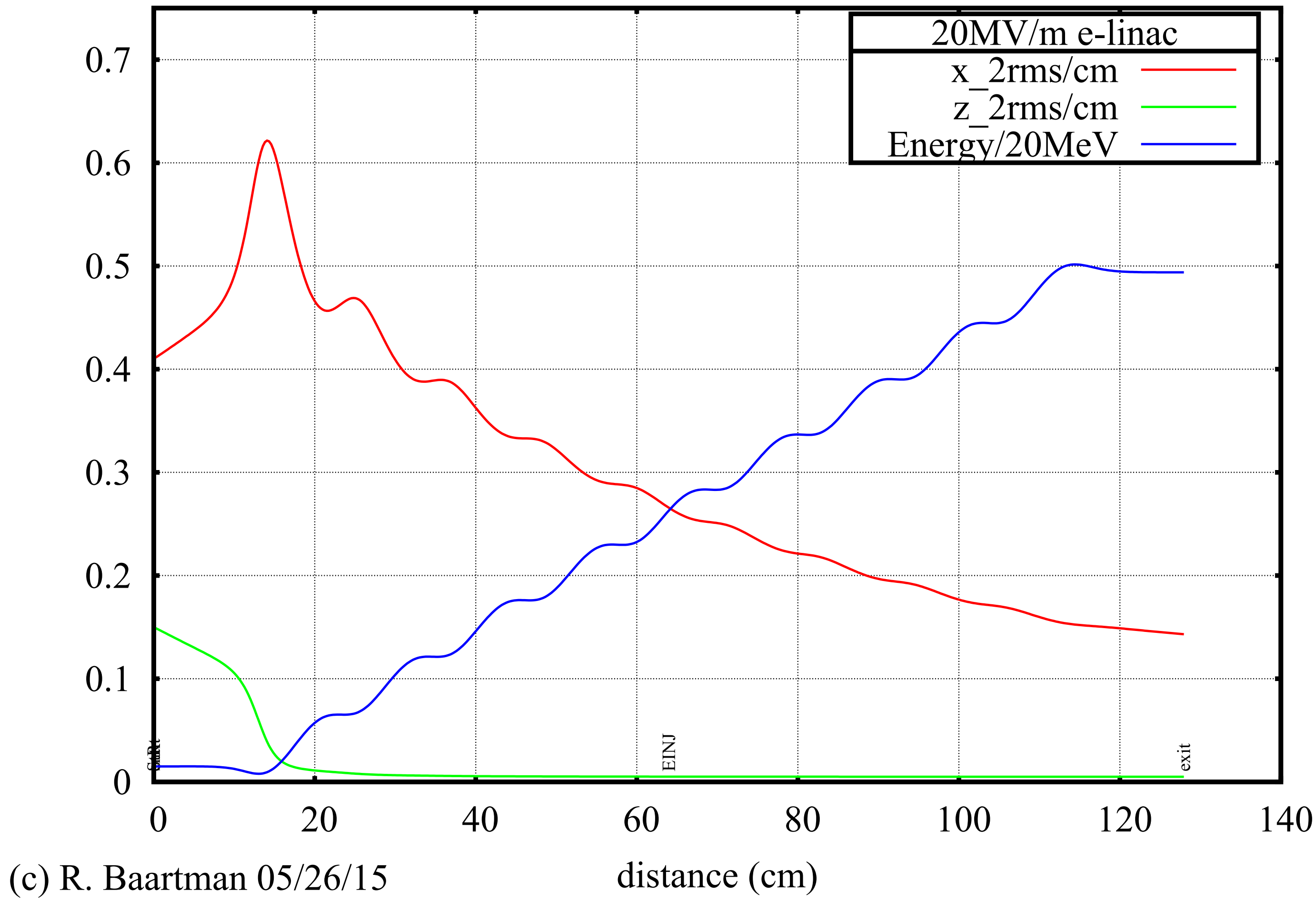
Example Calculations

The TRIUMF injector electron linac, EINJ, takes bunches at 300 keV to ~ 10 MeV if properly phased and the peak gradient is 20 MV/m. Here is the input $\mathcal{E}(s)$. TRANSOPTR interpolates with cubic splines.

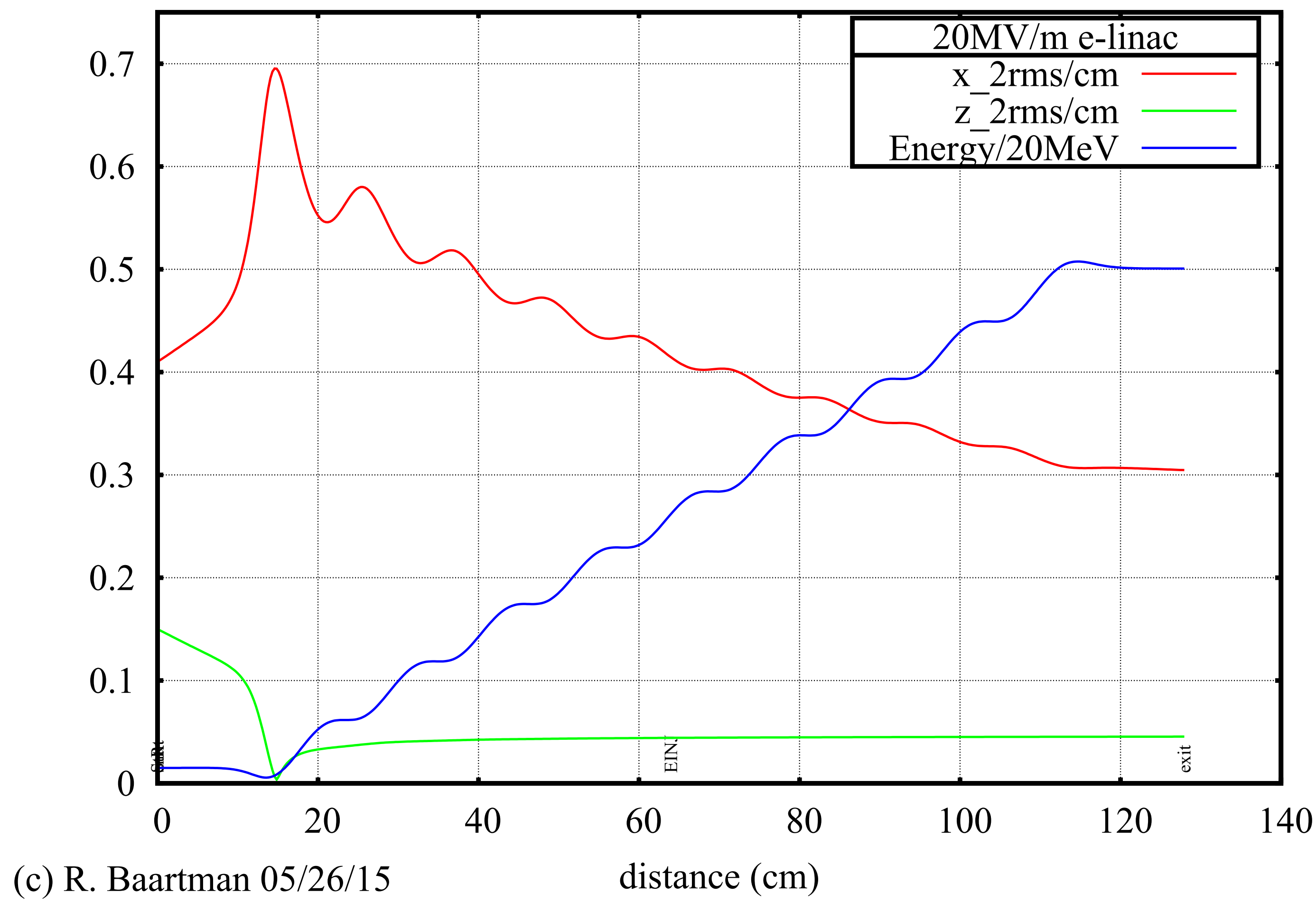


This is example for phase $\theta = 0$ at the start of the calculation.

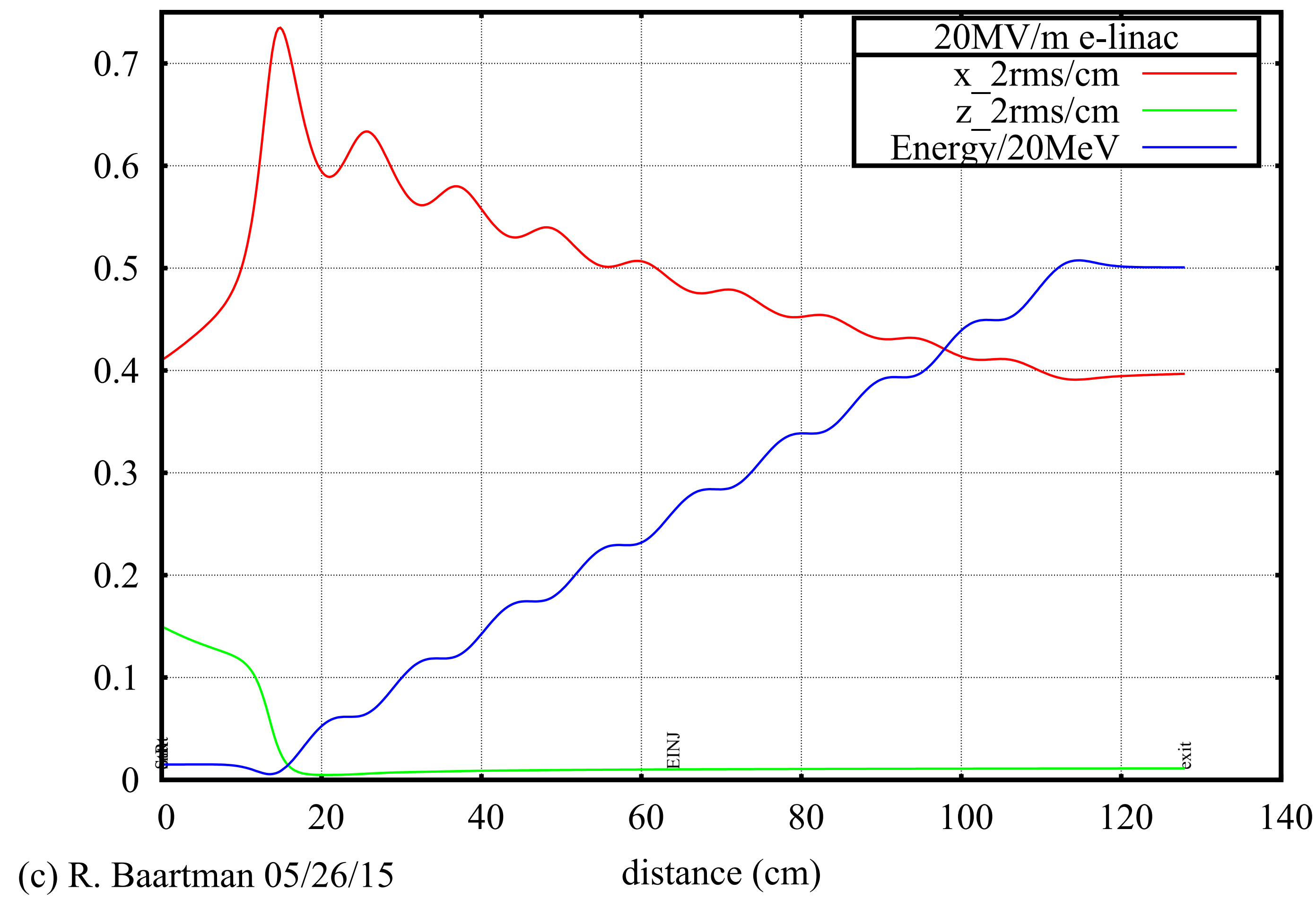
Red is the 2rms transverse size, and green is the 2rms longitudinal (bunch length). The input bunch parameters are somewhat arbitrary, roughly the condition for a minimum beam size at exit. This particular case has zero bunch charge.



In this second example, `TRANSOPTR` is instructed to fit the 65 matrix element to zero. This makes energy insensitive to input phase, thus finding the peak energy gain phase. This phase turns out to be $\theta = -15.46^{\circ}$.



In the third example, bunch charge has been raised to 30 pC.



Compute Efficiency

Each calculation above takes roughly 400 Runge-Kutta steps for 2400 calls to the `SCLINAC` routine. This gives 5-figure accuracy to the transfer matrix and the σ-matrix, and is easily enough for describing reality considering that the on-axis field is only known to 2 or 3 significant figures. The extra accuracy is useful, however for fitting matrix or beam matching, which is done with a downhill simplex method, or simulated annealing for cases of more than 3 fitting parameters.

On my unremarkable, circa 2006 Intel desktop, each run through the linac takes about 17 milliseconds with zero bunch charge and 25 milliseconds with space charge. The difference is due to the Carlson elliptic integrals needed for the space charge case.

apology: In fact, the speed could be 3 times faster. Instead of 23 ODEs, we solve 78. Lazily, I directly solve the σ equation of motion, and that's 36; not making use of the symmetry. As well, solve for all 36 components of the incoherent transfer matrix $\mathbf{M} = \int \mathbf{F} ds$. Lastly, solve for all $(x_0, P_{x0}, y_0, P_{y0}, t_0, E_0)$, not just the last two.

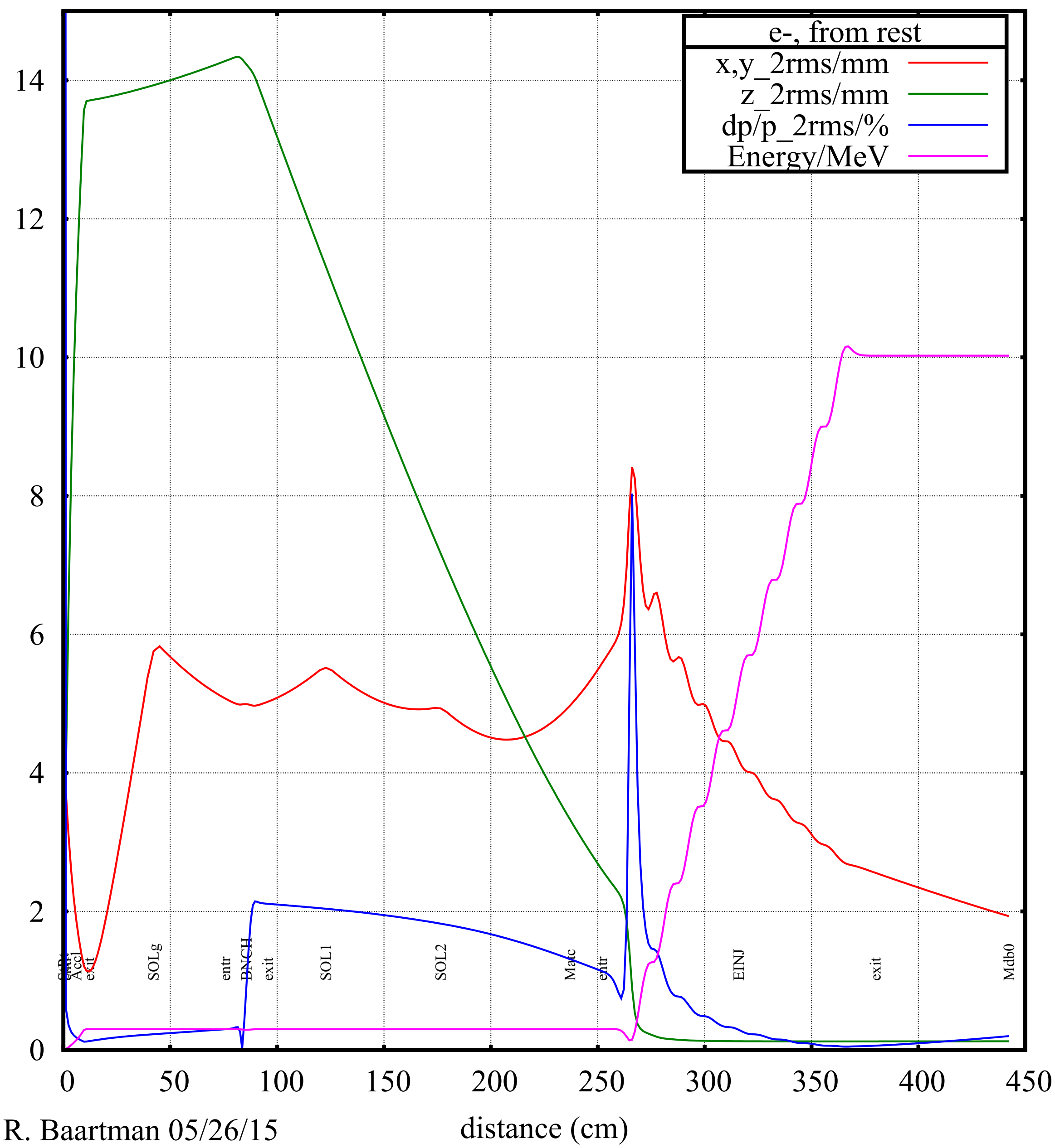
OTOH, matrix $\mathbf{M}$ by itself of course has uses. For example: Adjusting linac phase to get $M_{65} = 0$ finds the Peak Energy Gain phase.

On a typical optics matching case, one varies 2 solenoids, the buncher amplitude, and the linac phase, to minimize the bunch size and energy spread at the linac output. A calculation with such a fit requires typically a half million total calls to `SC` (the space charge routine for no-linac case) and `SCLINAC`, and so takes about 5 seconds CPU time. The result is shown below.

Each calculation starts from the cathode (it would have been more efficient to store the beam parameter set at the buncher entrance and start it from there). The bunch charge is 15 pC.

The Buncher itself, located at $s = 85$ cm, is calculated as just another linac, phased to give no energy gain.

Below is the GUI an operator would use. See Paul Jung, TRI-BN-15-13. Or, direct link to GUI.




ELINAC Beam Envelopes
beamphys.triumf.ca/projects/BeamEnvelope/BeamEnvelope-ELINAC.html
Rick
Tune
Expert
Console
Save
Load
Clear
Reset
egun-embd
Beam Envelopes Settings
Initial Beam
Energy 0.02 [keV]
Mass 0 [MeV]
Scale
Space Charge
Bunch Charge 1.5e-11 [C]
Load Tunes
Default
Optics
Name Value Optimize?
EGUN:SOL1 2.868 [A] 0 5
ELBT:BUN 0.477 [MV/m] 0.4 2
BUN:PH 45.0035 [deg] 45 135
ELBT:SOL1 3.9878 [A] 0 5
ELBT:SOL2 3.1506 [A] 0 5
ELBT:CAV 24.5 [MV/m] 0 30
ELBT:PH -51.8506 [deg] -180 180
EMBT:Q1 2.5583 [A] -5 5
EMBT:Q2 -4.9976 [A] -5 5
Update
Optimize
Close
s/m
Element
xEnvelope/cm
yEnvelope/cm
zEnvelope/cm
xDispersion/cm
dp/p %
Sigma16
Energy/(10*MeV)

Conclusions

Envelope calculations are typically 5 orders of magnitude faster than multiparticle space charge simulations.

You can try example above at
`http://beamphys.triumf.ca/projects/BeamEnvelope/BeamEnvelope-ELINAC.html`

Code freely available at
`http://lin12.triumf.ca/optr/src`

Acknowledgements: Paul Jung, Thomas Planche, Suresh Saminathan, Marco Marchetto, Brandon Humphries, Dobrin Kaltchev