EMITTANCE GROWTH FROM BEND/STRAIGHT TRANSITIONS FOR BEAMS APPROACHING THERMAL EQUILIBRIUM*

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Abstract

In certain applications such as heavy ion fusion, intense beams with large space charge tune depressions will be transferred from linear transport sections into bent transport sections. In some designs, such as recirculating induction accelerators, transport through bends will occur over thousands of betatron periods and in some driver designs the final transport through a bend will occur over tens of betatron periods. Over such distances, non-linear space charge forces are expected to produce particle phase space distributions which are close to thermal equilibrium, especially with respect to lower order moments. Here we calculate the properties of thermal equilibrium beams in bends assuming uniform focusing, as a function of two dimensionless parameters We also outline the calculation of the change in emittance for a beam that is initially in thermal equilibrium in a straight transport section, and that finally reaches thermal equilibrium in a bent system, using an energy conservation constraint to connect the two states.

1 INTRODUCTION

The conditions for equilibria of beams in a bent system were determined in ref. [1], under the assumption of uniform focusing and bending, with dispersion included through linear order in the equations of motion. The equilibria were determined by requiring that the derivatives of the second order moments with respect to path length vanish. A further assumption of this calculation was that space charge was distributed uniformly in an elliptical cross section, although as pointed out in refs. [2,3], distributions that are functions only of $x^2/\langle x^2\rangle + y^2/\langle y^2\rangle$ are also exact solutions to the moment equations of ref. [1], where x is the coordinate in the bend plane, y is the out-of-plane coordinate, and $\langle\ \rangle$ indicates average over the distribution.

Recently, in refs. [4,5] equilibrium distributions have been calculated that are fully self-consistent solutions to the coupled Vlassov and Poisson equations. Distribution functions which are functions only of the single particle transverse hamiltonian $h_\perp$ are solutions of the Vlasov equation, since $h_\perp$ is a constant of the motion. In refs. [4,5], the properties of a generalized KV distribution, (i.e. a delta-function of $h_\perp$) were investigated in detail. Although, the KV beam in bends is interesting because of its analytic tractability, beams which have equilibrated (e.g.due to space charge non-linearities), are likely to be better characterized by thermal equilibrium distributions. Although longitudinal/transverse coupling can be strong [6], it is of general interest to examine beams with distinct temperatures in the two directions.

The purpose of this paper, is to examine thermal equilibrium beams in bends with longitudinal temperatures which are not necessarily equal to the transverse temperatures (and hence the final temperature equilibration has not necessarily been reached.)

2 THEORETICAL MODEL

Equilibrium distribution functions f which satisfy the Vlassov/Poisson equations, for a system with constant focusing and bending radius have been found previously having the following form (refs. [4,5]):

$$f = f(h_\perp)\exp[-(\delta/\delta_0)^2] \tag{1}$$

where $2h_\perp = p_x^2 + p_y^2 + k_{\beta 0}^2(x^2+y^2) + 2g\phi - 2x\delta/\rho$. Here $f \equiv \mathrm{d}N/\mathrm{d}x\mathrm{d}y\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}\delta$, is the number of particles per element of phase space, with the in-bend plane (horizontal) coordinate x, and vertical coordinate y, dimensionless momenta $p_{x,y}$ normalized to the design momentum in in the longitudinal direction $P_0 \equiv \gamma_0 m v_0$. The quantity $\delta = (P_s - P_0)/P_0$, is the fractional deviation of a particle with longitudinal momentum P_s from the longitudinal design momentum, and m is the particle rest mass. The quantity $k_{\beta 0}$ is the zero current spatial betatron frequency in the postulated uniform focusing channel, and ρ is the radius of curvature in the uniform bending field. The quantity ϕ is the electrostatic potential, and $g \equiv q/\gamma_0^3 m v_0^2$.

In this paper, we focus on the distribution of the form:

$$f(x,y,p_x,p_y,\delta) = f_0\exp(-h_\perp/T_\perp)\exp(-\delta^2/\delta_0^2) \tag{2}$$

Here, $T_\perp \equiv k_b\mathcal{T}_\perp/\gamma_0^2 m v_0^2$ where $\mathcal{T}_\perp$ is the comoving beam transverse temperature, k_b is Boltzmann's constant. The density $n(x,y)$ is given by:

$$n(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,p_x,p_y,\delta)\,dp_x dp_y d\delta \tag{3}$$

$$= n(0,0)\exp\frac{-1}{T_\perp}\left(\frac{k_{\beta 0}^2}{2}([1-\eta]x^2+y^2) + g\Delta\phi(x,y)\right). \tag{4}$$

Here $\eta \equiv \delta_0^2/2k_{\beta 0}^2\rho^2 T_\perp$, and represents the effects of dispersion in a bend on off-momentum particles, and $\Delta\phi \equiv \phi(x,y) - \phi(0,0)$. We find solutions to the non-linear poisson's equation $\nabla^2\phi = -qn(x,y,\phi(x,y))/\epsilon_0$ for which the beam pipe (radius r_p) is sufficiently far from the beam edge such that image forces can be ignored.

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3 DIMENSIONLESS FORM OF MODEL

Without dispersion ($\eta = 0$) the beam density profiles in this problem recover azimuthal symmetry. In that case, the density profiles can be characterized by a single parameter (see e.g. [8]) which we define here as $\alpha_0 \equiv n(0,0)/n_{cold}$. Here $n(0,0)$ is the central density and n_{cold} is given by: $n_{cold} \equiv 2\gamma^3 m v_0^2 \epsilon_0 k_{\beta 0}^2/q^2$. The quantity n_{cold} is the density of a beam with focusing constant $k_{\beta 0}$, but at zero T_p and zero η.

When dispersion is added, the second dimensionless parameter η appears and all solutions may be characterized by the two dimensionless parameters α_0 and η. We define $X \equiv k_{\beta 0} x/T_\perp^{1/2}$, $Y \equiv k_{\beta 0} y/T_\perp^{1/2}$, $\Phi \equiv g\phi/T_\perp$, and $\Delta\Phi \equiv \Phi(X,Y) - \Phi(0,0)$. We may then cast Poisson's equation into the dimensionless form:

$$\frac{\partial^2 \Phi}{\partial^2 X^2} + \frac{\partial^2 \Phi}{\partial^2 X^2} = -2\alpha_0 \exp -\Psi. \qquad (5)$$

Here, $\Psi \equiv (1-\eta)X^2/2 + Y^2/2 + \Delta\Phi$. The boundary condition is $\Phi = 0$ at $X^2 + Y^2 = R_p^2$, where $R_p \equiv k_{\beta 0} r_p/T_\perp^{1/2}$ is the dimensionless pipe radius. Note that we have neglected the curvature terms in Poisson's equation, which is appropriate when $r_p << \rho$. Note also that R_p adds a third dimensionless parameter to the problem, but results presented here will be in a regime where R_p is large enough so that the beam parameters are nearly independent of R_p. We solve this equation numerically, using standard SOR techniques.

Once a solution is obtained, it is useful to calculate dimensionless moments of the density distribution: $I_1(\alpha_0, \eta) \equiv \int\int dX dY \exp -\Psi$;
$I_{X^2}(\alpha_0, \eta) \equiv \int\int dX dY\ X^2\ \exp -\Psi$;
$I_{Y^2}(\alpha_0, \eta) \equiv \int\int dX dY\ Y^2\ \exp -\Psi$;
and $I_\Phi(\alpha_0, \eta, R_p) \equiv \int\int dX dY\ \Phi\ \exp -\Psi$. Here,the integration occurs over the interior of the beam pipe, $X^2 + Y^2 < R_p^2$, and the explicit dependence on α_0 and η is displayed. From these quantities, averages can be obtained: $\langle X^2 \rangle \equiv I_{X^2}/I_1$, $\langle Y^2 \rangle \equiv I_{Y^2}/I_1$, and $\langle \Phi \rangle \equiv I_\Phi/I_1$.

Using these integrals and averages, which depend only on α_0 η, (and in the case of I_Φ, R_p), we may calculate physical parameters of the beam. For example, the current $I = q v_0 n_{cold} \alpha_0 T_\perp I_1/k_{\beta 0}^2$, the perveance $K \equiv qI/2\pi\epsilon_0 \gamma_0^3 m v_0^3 = \alpha_0 T_\perp I_1/\pi$, x-emittance $\epsilon_x = 4(\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2)^{1/2} = 4T_\perp (I_{X^2}/I_1)^{1/2}$, and space charge parameter $S = 4K\langle x^2 \rangle/\epsilon_x^2 = K/4T_\perp = \alpha_0 I_1/4\pi$. The rms tune depression $\sigma_x/\sigma_0 = (1/\langle X^2 \rangle + \eta)^{1/2}$, and $\sigma_y/\sigma_0 = 1/\langle Y^2 \rangle^{1/2}$.

4 RESULTS

Figure 1 displays a surface plot of the normalized beam density with a relatively large dispersion, and moderate tune depression. The beam has an apparent elliptical shape with a flattop similar to the thermal equilibrium beams in straight transport sections (cf. [8]).

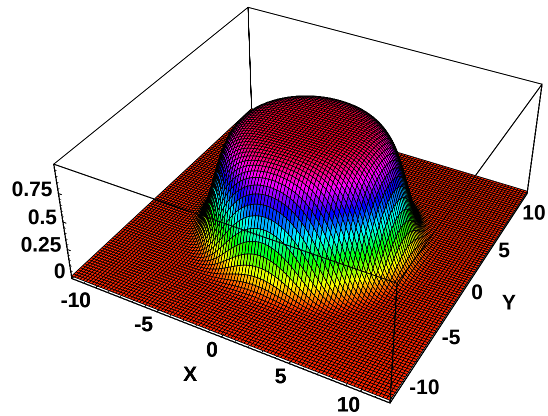

Figure 1. Scaled density $n(X,Y)/n_{cold}$ vs. X and Y for the parameters $\alpha_0 = 0.974$, and $\eta = 0.05$.

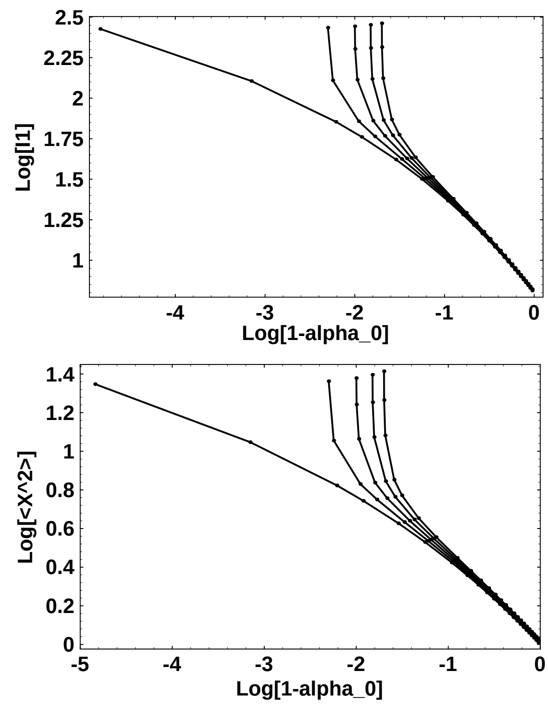

Figure 2. $\log(I_1)$ vs. $\log(1-\alpha_0)$ (upper) and $\log\langle X^2 \rangle$ vs. $\log(1-\alpha_0)$ (lower) for five different values of η (starting from the left-most curve and proceeding to the right, $\eta =$ 0.00, 0.01, 0.02, 0.03, and 0.04).

The curves asymptote to $\alpha_0 = 1 - \eta/2$ for large space charge depressions (derivable from the envelope equations below with zero emittance), and I_1 tends to $2\pi/\sqrt{1-\eta}$, while $\langle X^2 \rangle$ tends to $1/(1-\eta)$ in the limit of zero space charge.

5 EQUILIBRIUM EQUATIONS

In ref. [1], moment equations including dispersion were derived, and in ref. [9], the effects of images on a uniform density elliptical beam in a circular pipe were derived. The envelope equations with these two effects included (in addition to the usual external focusing, space charge and emittance terms are):

$$\frac{\mathrm{d}^2 a}{\mathrm{d}s^2} = -k_{\beta 0}^2 a + \frac{\epsilon_x^2}{a^3} + \frac{4}{\rho}\langle x\delta \rangle + \frac{2K}{a+b} + K\frac{(a^2-b^2)a}{4r_p^4} \qquad (6)$$

$$\frac{\mathrm{d}^2 b}{\mathrm{d}s^2} = -k_{\beta 0}^2 b + \frac{\epsilon_y^2}{b^3} + \frac{2K}{a+b} - K\frac{(a^2-b^2)b}{4r_p^4} \tag{7}$$

Here $a \equiv 2\sqrt{\langle x^2\rangle}$ and $b \equiv 2\sqrt{\langle y^2\rangle}$. Setting $\mathrm{d}^2a/\mathrm{d}s^2 = \mathrm{d}^2b/\mathrm{d}s^2 = 0$, and transforming to the dimensionless variables, we find the equilibrium moments satisfy:

$$0 = -(1-\eta)\sqrt{\langle X^2\rangle} + \frac{1}{\sqrt{\langle X^2\rangle}} + \frac{\alpha_0 I_1}{2\pi\left(\sqrt{\langle X^2\rangle}+\sqrt{\langle Y^2\rangle}\right)} + \frac{\alpha_0 I_1\left(\langle X^2\rangle-\langle Y^2\rangle\right)}{\pi R_p^4}$$

$$0 = -\sqrt{\langle Y^2\rangle} + \frac{1}{\sqrt{\langle Y^2\rangle}} + \frac{\alpha_0 I_1}{2\pi\left(\sqrt{\langle X^2\rangle}+\sqrt{\langle Y^2\rangle}\right)} - \frac{\alpha_0 I_1\left(\langle X^2\rangle-\langle Y^2\rangle\right)}{\pi R_p^4}$$

It has been found that in all cases examined, that given I_1, and solving for $\langle X^2\rangle$ and $\langle Y^2\rangle$, these equilibrium equations accurately predict the moments derived from the SOR code, and the final term accurately gives an indication of the importance of image charge effects on the solution.

6 EMITTANCE GROWTH FROM BEND/STRAIGHT TRANSITIONS

As discussed in [1], if a beam abruptly enters a bend from a straight transport section, off momentum particles will tend to oscillate in x about centers which are displaced from the design orbit of the machine. This causes an envelope mismatch, and if the non-linear space charge forces are sufficiently strong to allow phase mixing and energy equi-partition between the x and y directions, then a new equilibrium will result. In ref. [1], the moment equations yield an exact energy invariant, when $k_{\beta 0}$ is independent of s, under the assumption that density is constant on nested ellipses ($n(x,y) = n(x^2/\langle x^2\rangle + y^2/\langle y^2\rangle)$). More generally, a dimensionless average transverse energy may be written:

$$H_\perp = \frac{1}{2}\left((1-2\eta)\langle X^2\rangle + \langle Y^2\rangle + \langle\Phi\rangle + 2\right) \tag{8}$$

Because of the choice of normalization, it is the quantity $H_\perp T_\perp$ which is conserved. Note that the factor of 1/2 multiplying $\langle\Phi\rangle$ is necessary to correctly calculate the self-assembly energy from space charge. To calculate the change in beam parameters from a straight/bend transition, we first calculate the current I and the transverse energy $H_\perp T_\perp$ of the beam in the straight section. Because we tabulate $H_\perp(\alpha_0,\eta,R_p)$ for fixed R_p we must account for the change in R_p as $T_\perp$ changes even though r_p remains fixed. But $H_\perp(\alpha_{0f},\eta_f,R_{pf}) = H_\perp(\alpha_{0f},\eta_f,R_{pi}) + (K/2T_{\perp f})\times(\ln R_{pf} - \ln R_{pi})$, where subscripts i,f indicate initial, final. For $k_{\beta 0}$ and r_p held constant, we find $T_{\perp i}[H_\perp(\alpha_{0i},\eta=0,R_{pi}) + (K/4T_{\perp i})\ln K/T_{\perp i}] = T_{\perp f}[H_\perp(\alpha_{0f},\eta_f,R_{pi}) + (K/4T_{\perp f})\ln K/T_{\perp f}]$. For a finite value of η, we iterate $T_\perp$ and α_0, until the current and this relation for $H_\perp$ is satisfied. This allows calculation of all final beam parameters and using $\langle\delta^2\rangle = (1+\eta\langle X^2\rangle)\delta_0^2/2$, we may a posteriori, determine the initial value of δ_0. The change in emittance calculated using this algorithm agrees within numerical accuracy to the calculation done using the moment equations in ref. [1] and compared with simulations in ref. [7].

7 CONCLUSIONS

We have solved the self-consistent Vlasov Poisson system for beams in bends with thermal distributions, and with temperatures not necessarily equal in the longitudinal and transverse directions. We have characterized these beams by two dimensionless parameters α_0 and η and have graphed two of the quantities which characterize the solutions. We find that such beams have profiles which are constant on nested ellipses, to within numerical errors when the beam pipe is sufficiently large. This validates moment and envelope equations in refs. [1] and [3] for this class of beams. Emittance growth from bend/straight transitions, using energy and current conservation constraints was found to be the same as that calculated in ref. [1] again to within numerical uncertainties.

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