COUPLING IMPEDANCE IN AN ELLIPTICAL BEAM PIPE

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Introduction

For an ultrarelativistic ($\gamma \gg 1$) particle traveling in a beam pipe of constant cross-section, the calculation of the longitudinal/transverse coupling impedance reduces to two-dimensional calculation of the static fields due to a monopole/dipole charge or current singularity along the axis. In this paper, we formulate the general calculation of coupling impedance and apply it to an elliptical beam pipe. In particular, we obtain the longitudinal/transverse resistive wall impedance, as well as the longitudinal/transverse impedance of one or more small holes in the beam pipe.

Longitudinal Coupling Impedance

For a drive beam of current density

$$J_z = I_0\delta(x-x_1)\delta(y-y_1)\exp(-jkz) \tag{1}$$

in the frequency domain, with $k=\omega/c$, the longitudinal impedance is defined as

$$Z_{\parallel}(k) = -\frac{1}{I_0}\int_{-\infty}^{\infty} dz\ E_z e^{jkz}, \tag{2}$$

where E_z is the longitudinal component of the electric field when $x_1=0, y_1=0$. We use Eq. (1) to rewrite $Z_{\parallel}(k)$ as

$$Z_{\parallel}(k) = -\frac{1}{|I_0|^2}\int dv\ \vec{E}\cdot\vec{J}^*, \tag{3}$$

where the volume integral is over a region which includes the drive beam.

We now consider two situations. The first, denoted by the subscript 1, is the lossless pipe, and the second, denoted by the subscript 2, is the pipe with wall losses. We then construct

$$\begin{aligned}|I_0|^2[Z_{\parallel}^{(2)}(k) + Z_{\parallel}^{(1)^*}(k)] &= |I_0|^2[Z_{\parallel}^{(2)}(k) - Z_{\parallel}^{(1)}(k)] \\ &= -\int dv[\vec{E}_2\cdot\vec{J}^* + \vec{E}_1^*\cdot\vec{J}]. \end{aligned}\tag{4}$$

where $Z_{\parallel}^{(1)}(k)$ is imaginary. (It actually vanishes in the ultrarelativistic limit.) Using

$$\vec{J} = \nabla\times\vec{H}_{1,2} - j\omega\epsilon\vec{E}_{1,2}\ ,\quad \nabla\times\vec{E}_{1,2} = -j\omega\mu\vec{H}_{1,2}\ , \tag{5}$$

Eq. (4) can be converted into a surface integral, leading to

$$|I_0|^2 Z_{\parallel}(k) = \int dS\ \vec{n}\cdot[\vec{E}_2\times\vec{H}_1^* + \vec{E}_1^*\times\vec{H}_2], \tag{6}$$

where the surface encloses the drive beam. If we choose S to be the inside surface of the beam pipe, $\vec{n}\cdot\vec{E}_1^*\times\vec{H}_2 = 0$, and we have, for a length of beam pipe L,

$$|I_0|^2 Z_{\parallel}(k) = -L\oint ds E_z H_{1s}^*, \tag{7}$$

where s is a coordinate tangential to the beam pipe surface in a plane perpendicular to the axis of the beam pipe. The form in Eq. (7) is a generalization of a result derived earlier[1] for a beam pipe of circular cross-section and used recently by Napoly[2].

We now obtain the result for a resistive wall by expressing E_z at the wall in terms of H_{1s}. Specifically we take

$$E_z \cong -k\delta(1+j)Z_0 H_{1s}/2, \tag{8}$$

where $\delta = (2/k\sigma Z_0)^{1/2}$ is the skin depth of the wall material whose conductivity is σ. Here $Z_0 = (\mu/\epsilon)^{1/2} \cong 120\pi$ ohms is the impedance of free space. Using Eq. (8), we write the longitudinal impedance as

$$|I_0|^2 Z_{\parallel}(k)/Z_0 = (1+j)(kL\delta/2)\oint ds|H_{1s}|^2. \tag{9}$$

Finally, H_{1s} can be obtained from the solution of the Laplace (or Poisson) equation in the two transverse dimensions since $c^2\partial^2/\partial^2 z^2 = \partial^2/\partial t^2$ for an ultrarelativistic particle. Specifically

$$Z_0 H_{1s} = E_{1n} = -\exp(-jkz)\nabla_\perp\Phi(x,y), \tag{10}$$

where $\Phi(x,y)$ is the solution of

$$\nabla_\perp^2\Phi(x,y) = -Z_0 I_0\delta(x-x_1)\delta(y-y_1). \tag{11}$$

with perfectly conducting boundary conditions at the beam pipe wall. Here n is a coordinate normal to the beam pipe wall and E_{1n} is the electric field normal to the beam pipe surface for the lossless problem.

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Transverse Coupling Impedance

The transverse coupling impedance can be analyzed in a similar manner. If we start with the axial dipole drive current

$$J_z = I_0\delta(y)\exp(-jkz)[\delta(x-x_1)-\delta(x+x_1)], \qquad (12)$$

the transverse impedance in the x-direction can be expressed as the limit for small x_1 of

$$Z_x(k) = -\frac{1}{2kI_0x_1}\int_{-\infty}^{\infty} dz\frac{\partial E_z}{\partial x}e^{jkz}, \qquad (13)$$

where $\partial E_z/\partial x$ is evaluated for $x=y=0$. But we can also write the derivative of E_z at the origin as

$$\frac{\partial E_z}{\partial x} = \frac{E_z(x_1,0,z)-E_z(-x_1,0,z)}{2x_1} \qquad (14)$$

for vanishingly small x_1. Thus we have

$$\begin{aligned} Z_x(k) = & -\frac{1}{4kI_0x_1^2}\int_{-\infty}^{\infty} dz[E_z(x_1,0,z) \\ & - E_z(-x_1,0,z)]e^{jkz}. \end{aligned} \qquad (15)$$

Using the value of $\vec{J}$ in Eq. (12), we can therefore write

$$Z_x(k) = -\frac{1}{4kx_1^2|I_0|^2}\int dv\vec{E}\cdot\vec{J}^*, \qquad (16)$$

in analogy with Eq. (3). As before, the volume integral in Eq. (16) can be written as a surface integral, and we obtain

$$4x_1^2|I_0|^2kZ_x(k) = -L\oint dsE_zH_{1s}^*, \qquad (17)$$

where we must now use the fields corresponding to the dipole configuration in Eq. (12). Finally, we use Eq. (8) to obtain

$$4x_1^2|I_0|^2Z_x(k)/Z_0 = (1+j)(L\delta/2)\oint ds|H_{1s}|^2. \qquad (18)$$

Beam Pipe of Elliptical Cross Section

The Poisson equation for the electrostatic potential of a line charge of density λ located at $x=x_1, y=y_1$ is

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = -\frac{\lambda}{\epsilon_0}\delta(x-x_1)\delta(y-y_1), \qquad (19)$$

where λ/ϵ_0 can be written in terms of the drive currnt as $\lambda/\epsilon_0 = Z_0I_0$. We transform to elliptic coordinates defined by

$$x = c\cosh u\cos v \qquad (20)$$

$$y = c\sinh u\sin v \qquad (21)$$

where the beam pipe is an ellipse of major axis $2a$, minor axis $2b$, with

$$a = c\cosh u_0\ ,\ b = c\sinh u_0\ ,\ c^2 = a^2-b^2. \qquad (22)$$

In the transformed coordinate system, Eq. (19) becomes

$$\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2} = -Z_0I_0\delta(u-u_1)\delta(v-v_1), \qquad (23)$$

where u_1, v_1 are related to x_1, y_1 by Eqs. (20) and (21).

We write the solution to Eq. (23) as[3]

$$\Phi(u,v) = f_0(u)+\sum_{n=1}^{\infty} f_n(u)\cos nv+\sum_{n=1}^{\infty} g_n(u)\sin nv, \quad (24)$$

where $f_n(u)$ turns out to be proportional to $\cosh nu\cos nv_1$ and $g_n(u)$ turns out to be proportional to $\sinh nu\sin nv_1$. For the longitudinal impedance, we use

$$|H_{1s}| = -\frac{1}{Z_0h}\frac{\partial\Phi}{\partial u} = \frac{I_0}{2\pi}\frac{Q_0(v)}{h(v)} \qquad (25)$$

where the metric $h(v)$ is given by

$$h(v) = c(\sinh^2 u_0+\sin^2 v)^{1/2}, \qquad (26)$$

and where

$$Q_0(v) \equiv 1+2\sum_{m=1}^{\infty}(-1)^m\frac{\cos 2mv}{\cosh 2mu_0}. \qquad (27)$$

In this way find

$$\frac{Z_\parallel(k)}{nZ_0} = \frac{(1+j)\delta}{2b}G_0(u_0), \qquad (28)$$

where $n = kL/2\pi$ is the harmonic number, and where

$$G_0(u_0) = \frac{\sinh u_0}{2\pi}\int_0^{2\pi}\frac{Q_0^2(v)dv}{[\sinh^2 u_0+\sin^2 v]^{1/2}}. \qquad (29)$$

In a similar way we obtain the transverse impedance

$$\frac{Z_{1x,y}(k)}{Z_0} = \frac{L(1+j)\delta}{2\pi b^3}G_{1x,y}(u_0), \qquad (30)$$

where

$$G_{1x,y}(u_0) = \frac{\sinh^3 u_0}{4\pi}\int_0^{2\pi}\frac{Q_{1x,y}^2(v)dv}{[\sinh^2 u_0+\sin^2 v]^{1/2}}. \qquad (31)$$

In this case we have

$$Q_{1x}(v) = 2\sum_{m=0}^{\infty}(-1)^m(2m+1)\frac{\cos(2m+1)v}{\cosh(2m+1)u_0}, \qquad (32)$$

$$Q_{1y}(v) = 2\sum_{m=0}^{\infty}(-1)^m(2m+1)\frac{\sin(2m+1)v}{\sinh(2m+1)u_0}. \qquad (33)$$

We have chosen a normalization such that $G_0(\infty) = G_{1x}(\infty) = G_{1y}(\infty) = 1$, reproducing the well known results for a circular beam pipe.

A graph of the numerical values of G_0, G_{1x}, G_{1y} is presented in Fig. 1 as a function of $q = (a-b)/(a+b)$. The values for $q=1$ correspond to parallel plates, and are in agreement with results obtained by others.

Coupling Impedance of Holes in the Beam Pipe

We start with Eqs. (7) and (17) and assume that the dimensions of the hole are small compared with the wavelength. In this case, the coupling integral

$$L \oint ds E_z H_{1s}^* = \int dS \, \vec{n} \cdot \vec{E} \times \vec{H}_1, \qquad (34)$$

written here as an integral over the interior aperture of the hole, can be expressed in terms of the inside electric polarizability, χ_{in}, and inside magnetic susceptibility, ψ_{in}, of the hole as

$$\int dS \, \vec{n} \cdot \vec{E} \times \vec{H}_1 = -j \frac{k|H_{1s}|^2}{2} (\psi_{in} - \chi_{in}). \qquad (35)$$

We have here assumed that the field outside the beam pipe can be ignored. A more complete discussion of the inside and outside polarizability and susceptibility is given elsewhere, including numerical results for a circular hole in a wall of finite thickness[4].

Once ψ_{in} and χ_{in} are known, the impedance can be calculated from $|H_{1s}|^2$ along the beam pipe wall. For the longitudinal coupling impedance, $|H_{1s}|$ is proportional to $Q_0(v)$ in Eq. (27) for an elliptical beam pipe, where v is the azimuthal coordinate of the hole. For the transverse coupling impedance the corresponding quantities are $Q_{1x,1y}(v)$ in Eqs. (32) and (33).The impedances of well separated holes (by at least a few hole diameters) can be added to each other, since the surface integral in Eq. (34) extends over all holes.

References

[1] R.L. Gluckstern and F. Neri, IEEE Transactions in Nuclear Science, Vol. NS-32, No. 5, 2403 (1985).

[2] O. Napoly, "The Wake Potentials from the Fields on the Cavity Boundary", Saclay Report CEA, DPhN/STAS/91-R12 (1991).

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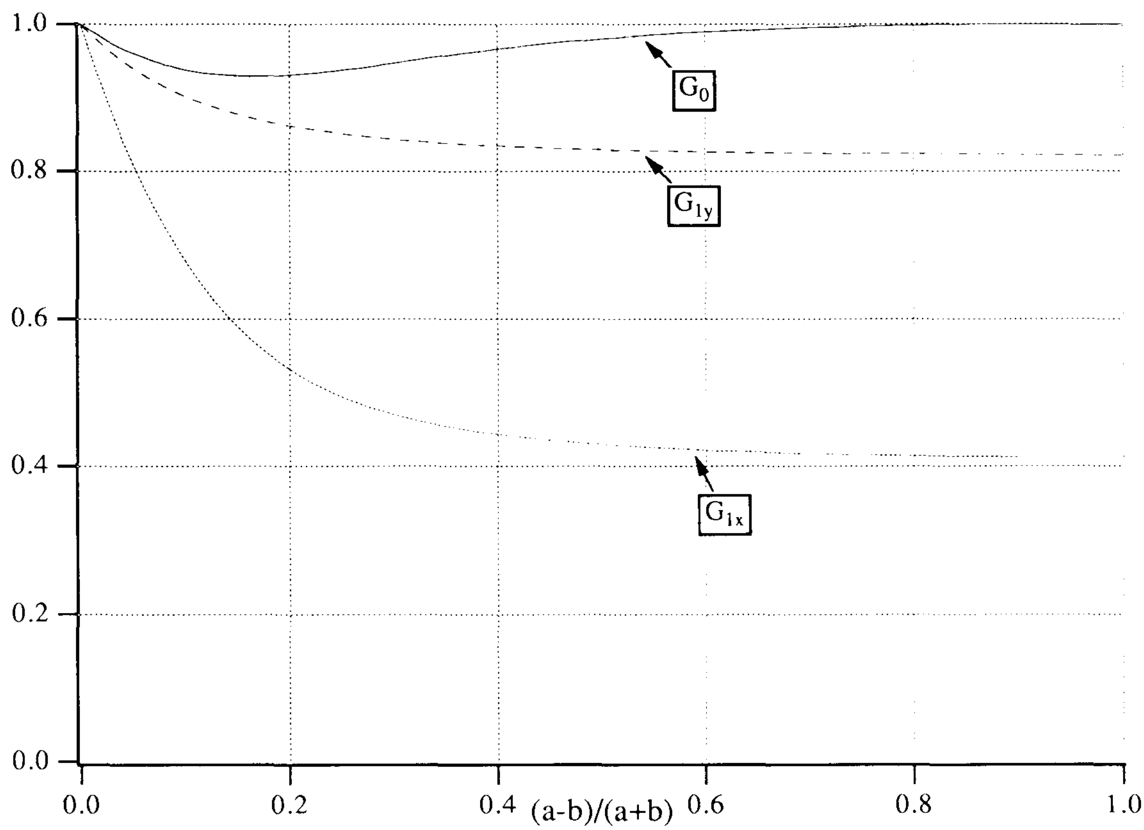

Figure 1: numerical values of $G_0(q), G_{1x}(q)$ and $G_{1y}(q)$ for the elliptical pipe as a function of the "nome" $q = (a-b)/(a+b)$.