LONGITUDINAL COUPLING IMPEDANCE OF A THICK IRIS COLLIMATOR

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Introduction

In a previous note[1] a method was presented for calculating the longitudinal coupling impedance of an iris in a beam pipe. In particular we considered a point charge Q traveling along the z-axis of a beam pipe of radius a, obtaining the source fields in the frequency domain (time dependence $\exp(j\omega t)$):

$$E_r^{(s)}(r,z;k) = Z_0^{(s)} H_\theta(r,z;k) = \frac{Q}{2\pi r} e^{-jkz}, \quad E_z^{(s)} = 0. \quad (1)$$

We solved two separate problems, each with a simplifying symmetry, by separating the source fields into a part even in z ($\cos kz$) and odd in $z(-j\sin kz)$. The even and odd problems were then solved by writing the fields as a sum of outgoing TM_{on} modes in the waveguide region $|z| \geq g/2$ and either symmetric or antisymmetric waveguide modes in the iris region $|z| \leq g/2, r \leq b$. Matching the boundary conditions at $z = \pm g/2$ then led to an infinite set of linear equations for the mode coefficients, and solutions were then obtained by truncating the resulting matrix equations.

The numerical work which then followed turned out not to be well convergent. In the present work we construct an alternate basis vector for the matrix equations, and find that the numerical implementation is well convergent.

In this paper we outline the new calculation and present numerical results for the limit $a/b \to \infty$, corresponding to a beam passing through a circular hole in a thick wall.

Matrix Equations

The field components for the even problem (E_r is even in z, E_z and H_θ are odd) are written in the waveguide region ($|z| \geq g/2, \ 0 \leq r \leq a$) as

$$E_r = \frac{Q}{2\pi r}\cos kz + \frac{Q}{2\pi}\sum_{m=1}^{\infty} A_m J_1\left(\frac{p_m r}{a}\right) e^{-j\beta_m(|z|-g/2)} \quad (2)$$

$$Z_0 H_\theta = -j\frac{Q}{2\pi r}\sin kz \pm \frac{Q}{2\pi}\sum_{m=1}^{\infty} \frac{kA_m}{\beta_m} J_1\left(\frac{p_m r}{a}\right) e^{-j\beta_m(|z|-g/2)}, \quad (3)$$

where p_m are the zeros of $J_0(p)$. Here

$$\beta_m = (k^2 - p_m^2/a^2)^{1/2} = -j(p_m^2/a^2 - k^2)^{1/2}, \quad (4)$$

where the sign of the second term in Eq. (4) is chosen so that the terms in the sums in Eqs. (2) and (3) correspond to outgoing waves in the beam pipe. Also, the $\pm$ sign corresponds to $z \gtrless 0$. In the iris region ($|z| \leq g/2, \ 0 \leq r \leq b$) we similarly write

$$E_r = \frac{Q}{2\pi r}\cos kz - \frac{Q}{2\pi}\sum_{n=1}^{\infty} B_n J_1\left(\frac{p_n r}{b}\right)\frac{\cos\sigma_n z}{\cos(\sigma_n g/2)}, \quad (5)$$

$$Z_0 H_\theta = \frac{-jQ}{2\pi r}\sin kz + \frac{jQ}{2\pi}\sum_{n=1}^{\infty}\frac{kB_n}{\sigma_n} J_1\left(\frac{p_n r}{b}\right)\frac{\sin\sigma_n z}{\cos(\sigma_n g/2)}, \quad (6)$$

where

$$\sigma_n = (k^2 - p_n^2/b^2)^{1/2}. \quad (7)$$

Matching $E_r(r, g/2)$ in the interval $0 \leq r \leq a$ leads to

$$A_m \frac{a^2}{2} J_1^2(p_m) = -\sum_{n=1}^{\infty} P_{mn} B_n - \frac{a}{p_m}\cos\frac{kg}{2} J_0\left(\frac{p_m b}{a}\right), \quad (8)$$

where

$$P_{mn} = \int_0^b r dr J_1\left(\frac{p_m r}{a}\right) J_1\left(\frac{p_n r}{b}\right) = \frac{(b/a)p_m J_1(p_n) J_0(p_m b/a)}{p_n^2/b^2 - p_m^2/a^2}. \quad (9)$$

Matching H_θ in the region $0 \leq r \leq b$ leads to

$$j\frac{B_n}{\sigma_n}\tan\frac{\sigma_n g}{2}\,\frac{b^2}{2} J_1^2(p_n) = \sum_{m=1}^{\infty}\frac{A_m}{\beta_m} P_{mn}. \quad (10)$$

Our task is to solve Eqs. (8) and (9) for A_m and B_n.

In the earlier report[1] we eliminated B_n and obtain a linear set of equations for A_m. We now instead eliminate A_m to obtain the matrix equation

$$\sum_{n'} M_{nn'} B_{n'} = -\cos\frac{kg}{2} f_n, \quad (11)$$

where

$$M_{nn'} = j\frac{a^2b^2J_1^2(p_n)\tan(\sigma_n g/2)}{4\sigma_n}\delta_{nn'} + \sum_{m=1}^{\infty}\frac{P_{mn}P_{mn'}}{\beta_m J_1^2(p_m)} \quad (12)$$

and

$$f_n = a\sum_{m=1}^{\infty}\frac{J_0(p_m b/a)P_{mn}}{\beta_m p_m J_1^2(p_m)}. \quad (13)$$

Note that the matrix **M** is symmetrix in $n \leftrightarrow n'$.

The contribution of this source term to the impedance can be written as an integral of the fields over the surface[2] at $z = \pm g/2$, $b \le r \le a$. Although this is directly obtained in terms of the coefficients A_m, we use Eq. (8) to express it in terms of B_n as $Z_{even} = Z_0(u_1 + u_2)$, where

$$u_1 = \frac{2k\cos^2(kg/2)}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(p_m b/a)}{\beta_m J_1^2(p_m)p_m^2} \quad (14)$$

and

$$u_2 = \frac{2k\cos(kg/2)}{\pi a^2}\sum_{n=1}^{\infty}B_n f_n. \quad (15)$$

The quantity u_1 can be explicitly calculated, and u_2 can be put into the form

$$u_2 = -\frac{2k\cos^2(kg/2)}{\pi a^2}\frac{[\sum_{n=1}^{\infty}B_n f_n]^2}{\sum_n\sum_{n'}B_n B_{n'}M_{nn'}}. \quad (16)$$

Equation (16) is now in a variational form with respect to the coefficients B_n, which means that requiring u_2 to be an extremum subject to the variation of f_n, leads to the matrix equation for u_2 in Eq. (15). For this reason, truncation of the matrix can be expected to lead to fairly accurate results for u_2. In fact, if we solve Eq. (11) for B_n by writing

$$B_n = -\cos\frac{kg}{2}\sum_{n'}(\mathbf{M}^{-1})_{nn'}f_{n'} \quad (17)$$

our final expression for u_2 becomes

$$u_2 = \frac{-2k\cos^2(kg/2)}{\pi a^2}\sum_n\sum_{n'}f_n(\mathbf{M}^{-1})_{nn'}f_{n'}. \quad (18)$$

A parallel calculation for the odd problem (E_r is odd in z, E_z and H_θ are even) leads to $Z_{odd} = Z_0(u_3 + u_4)$, where

$$u_3 = \frac{2k\sin^2(kg/2)}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(p_m b/a)}{\beta_m J_1^2(p_m)p_m^2}, \quad (19)$$

and

$$u_4 = \frac{-2k\sin^2(kg/2)}{\pi a^2}\sum_n\sum_{n'}f_n(\mathbf{N}^{-1})_{nn'}f_{n'}. \quad (20)$$

Here the matrix **N** is obtained from M by replacing $\tan(\sigma_n g/2)$ by $-\cot(\sigma_n g/2)$ in Eq. (12). Note that

$$u_1 + u_3 = \frac{2k}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(p_m b/a)}{\beta_m p_m^2 J_1^2(p_m)} \quad (21)$$

is independent of g.

Circular Hole in a Thick Wall Perpendicular to the Beam Axis

The result for a circular hole in a wall perpendicular to the beam axis can be obtained by proceeding to the limit $a \to \infty$ in Section 2. In this limit the main contribution to the sum over m comes from large m, and the sum over m can be converted to an integral over $x = p_m b/a$, with an interval $dx \cong \pi b/a$.

Let us start with $u_1 + u_3$ in Eq. (21) which can be written as

$$u_1 + u_3 = \frac{2kb}{\pi}\sum_m\frac{J_0^2(x_m)}{\sqrt{k^2b^2 - x_m^2}}\frac{1}{p_m^2 J_1^2(p_m)}, \quad (22)$$

where $x_m = p_m b/a$. In order to obtain convergence as $m \to \infty$, we rewrite Eq. (22) as

$$u_1 + u_3 = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(x_m)}{p_m^2 J_1^2(p_m)} + \frac{2kb}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(x_m)}{p_m^2 J_1^2(p_m)}\left[\frac{1}{\sqrt{k^2b^2 - x_m^2}} - \frac{1}{kb}\right]. \quad (23)$$

The first sum over m can be done explicitly and leads to

$$\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{J_0^2(x_m)}{p_m^2 J_1^2(p_m)} = \frac{1}{\pi}\ell n\left(\frac{a}{b}\right). \quad (24)$$

The second term in the sum in Eq. (23) now can be evaluated in the limit $m \to \infty$, leading to

$$u_1 + u_3 = \frac{1}{\pi}\ell n\left(\frac{a}{b}\right) + \frac{1}{\pi}\int_0^{\infty}\frac{dt}{t}J_0^2(kbt)\left[\frac{1}{\sqrt{1 - t^2}} - 1\right] \quad (25)$$

which converges satisfactorily at $t \to 0$ and at $t \to \infty$ as long as $kb \ne 0$. In fact, we can show that

$$u_1 + u_3 \to \frac{1}{\pi}[\ell nka + C] + \frac{j}{2}, \quad kb \to 0, \quad (26)$$

where $C = .577$ is Euler's constant, and

$$u_1 + u_3 \to j/\pi^2 kb, \quad kb \to \infty. \quad (27)$$

The calculation of u_2 and u_4 proceeds in a similar way, although here there are no divergent terms as $a \to \infty$.

One can show that u_2 and u_4 vanish when $kb \to 0$. This is confirmed in the numerical calculations where the result in Eq. (26) appears to be correct for $Z(k)/Z_0$ in the limit $kb \to 0, a/b \to \infty$, independent of g/b.

Numerical Results

We have constructed a numerical program which uses Eqs. (21), (18) and (20) to explore the dependence of Z/Z_0 on the three parameters $kb, g/b$ and a/b. Results for $g = 0$ were found to agree with results obtained earlier using a different analysis and computational procedure.[3]

In this paper we explore the limit $a/b \to \infty$ numerically. We find, in agreement with others[4], that the real part of the impedance becomes infinite. In our numerical work we therefore calculate $R' + jX = Z/Z_0 - (1/\pi)\ell n(a/b)$, which remains finite as $a/b \to \infty$. The validity of this approach is shown in Fig. 1 where R' and X are plotted against kb for $g = 0$ and for $a/b = 1000, 100, 25, 10$. For each value of a/b there are high frequency oscillations with phase ka. In our calculations the plotted points are an average over each such oscillation. Even for a/b as low as 10, R' and X differ from the infinite a/b result by less than .02.

In Fig. 2 we plot $R'' = \Re(Z/Z_0) - (1/\pi)\ell n(ka)$ and X for $g = 0$ in the low frequency region $0 < kb < 2$ for $a/b = 1000, 100$. The results confirm the limits of $R'' = C/\pi = 0.184$, and $X = 1/2$, implied by Eq. (26).

The dependence on g/b is shown in Figs. 3 and 4 where R' and X are plotted against kb for $a/b = 100$ and $g/b = 0, 0.2, 1, 5$. Clearly there is structure related to the value of g/b. But the results differ from the $g = 0$ smooth result by less than .05 over the entire range of kb and this difference decreases as kb increases.

References

[1] R.L. Gluckstern, "Longitudinal Coupling Impedance of an Iris in a Beam Pipe", CERN Report SL/90-113 (AP), October 1990.

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[3] R.L. Gluckstern and W.F. Detlefs, Longitudinal Coupling Impedance of a Thin Iris Collimator, Proceedings of the 1991 Particle Accelerator Conference, San Francisco, CA, May 1991, p. 1600.

[4] Dôme et. al., Il Nuovo Cinento A, 104, 8 (1991).

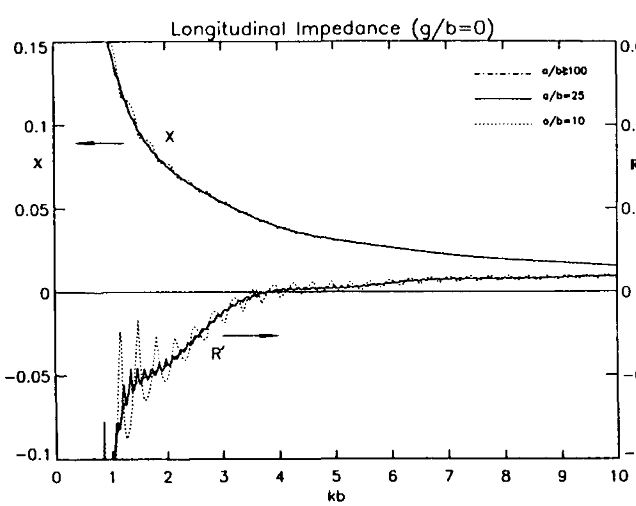

Fig. 1. R' and X vs. kb for $g/b = 0, a/b = 1000, 100, 25, 10$.

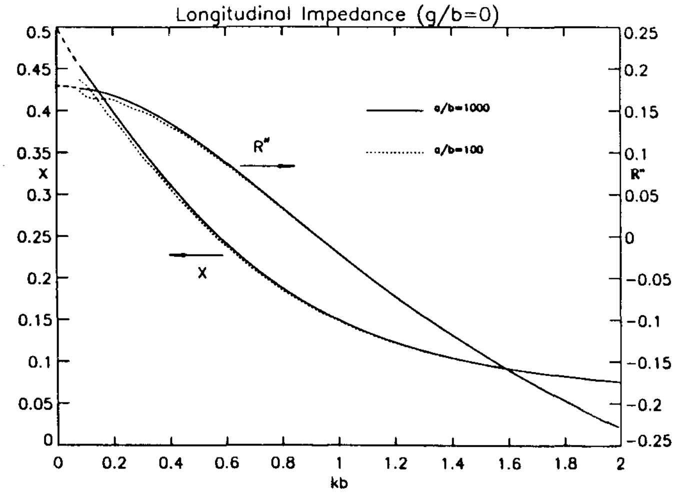

Fig. 2. R'' and X vs. kb for $g/b = 0, a/b = 1000, 100$.

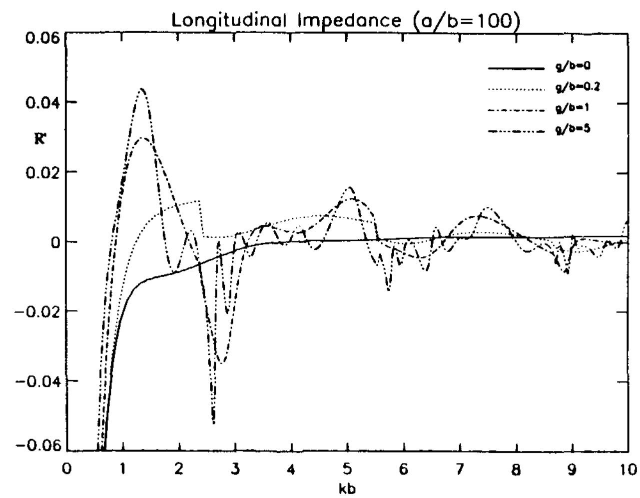

Fig. 3. R' vs. kb for $a/b = 100, g/b = 0, .2, 1, 5$.

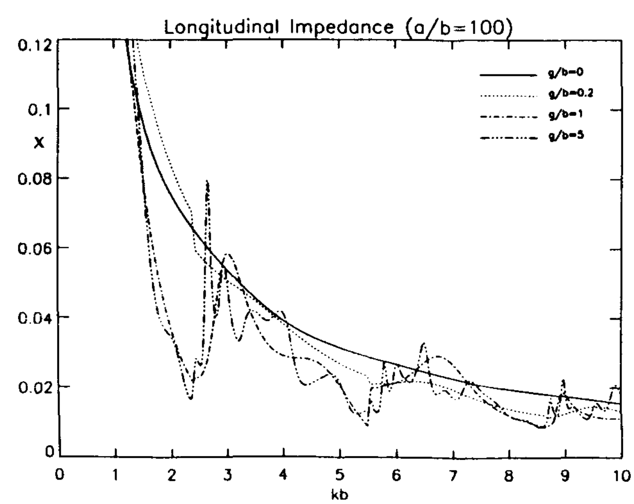

Fig. 4. X vs. kb for $a/b = 100, g/b = 0, .2, 1, 5$.