INFLUENCE OF A DISTRIBUTION OF DEFLECTING-MODE FREQUENCIES ON THE TRANSIENT DYNAMICS OF CUMULATIVE BEAM BREAKUP*

C. L. Bohn and J. R. Delayen
Engineering Physics Division, Argonne National Laboratory
9700 South Cass Avenue, Argonne, IL 60439 USA

Abstract

A distribution of deflecting-mode frequencies in the constituent cavities of a linear accelerator can lead to Q-independent damping of cumulative beam breakup. A probability density for the deflecting-mode frequencies generates an effective transverse wake function. The effective wake function can be used to calculate the transient dynamics of cumulative beam breakup within the framework of a continuum approximation provided the transverse beam displacement changes little over the correlation length of the deflecting-mode frequencies as the beam moves down the linac. We adopt this approach to show that the damping induced by the effective wake function causes the rate of approach to the steady state to depend strongly on the operative probability density for the deflecting-mode frequencies.

Introduction

In an earlier paper [1], hereafter called Paper 1, we developed an analytic formalism of cumulative beam breakup (CBBU) in linear accelerators with periodic beam current. The equation of transverse motion was expressed using a continuum approximation in which the discrete kicks in transverse momentum imparted by the cavities are considered to be smoothed along the linac, and it was solved by Fourier analysis. We used the formalism to study the role of a distribution of deflecting-mode frequencies on steady-state CBBU, showing that the principal effect is the replacement of the single-mode wake function by an effective wake function which depends strongly on the assumed probability density of the deflecting-mode frequencies. The problem of transient CBBU was not investigated in detail, but we observed that the effective wake function defined in the context of steady-state CBBU could also be used to study transient CBBU provided the transverse displacement does not change significantly over distances comparable to the correlation length of the deflecting-mode frequencies. Based on this observation, we reasoned that for deflecting modes with infinite Q the transient CBBU, and in particular the rate of approach to steady state, would also depend strongly on the assumed probability density of the deflecting-mode frequencies, and not just on the frequency spread.

Previous investigations support this reasoning. Colombant and Lau [2] find an algebraic decay to the steady state with a uniform probability density of deflecting-mode frequencies. By contrast, a Lorentzian probability density gives an exponential decay like that produced by a finite deflecting-mode Q [3]. Gluckstern, Neri, and Cooper [4,5] investigate transient CBBU for a beam of δ-function bunches in terms of the root-mean-square (rms) frequency spread for the case that the frequency spread is small compared to the frequency spread associated with Q. They find a decay which is faster than exponential. None of these investigators, however, explicitly identify the influence of the shape of the probability density on the rate of approach to the steady state.

In this paper we calculate explicitly the transient CBBU for both Gaussian and Lorentzian probability densities of deflecting-mode frequencies. These calculations illustrate how to apply our formalism given an arbitrary probability density. In the process, we confirm our earlier conjecture that the rate of approach to the steady state depends strongly on the shape of the probability density.

Equation of Transverse Motion

According to Paper 1, the equation of transverse motion for a coasting beam is

$$\left[\frac{d^2}{d\sigma^2}+\kappa^2\right]x(\sigma,\zeta) = \epsilon\int_0^{\zeta}d\zeta' w\left[\frac{\omega(\sigma)}{\omega_0}(\zeta-\zeta')\right]F(\zeta')x(\sigma,\zeta'). \quad (1)$$

Here, σ is the longitudinal coordinate normalized to the linac length; ϵ and κ represent the CBBU coupling strength and the focusing strength, respectively; $\omega(\sigma)$ is the angular frequency of the deflecting mode distributed according to the probability density $\mathcal{f}(\omega)$ around the mean frequency ω_0; $\zeta=\omega_0 t$ is the time normalized to ω_0; $F(\zeta)\equiv I(\zeta)/\bar{I}$ is the form factor defined in terms of the beam current $I(\zeta)$ and the average current $\bar{I}$; and $w(\zeta)$ is the wake function for the single deflecting mode:

$$w(\zeta) = u(\zeta)e^{-\zeta/2Q}\sin\zeta, \quad (2)$$

where $u(\zeta)$ is the unit step function.

We first average eq. (1) over a length $\Delta\sigma$ around σ:

$$\left[\frac{d^2}{d\sigma^2}+\kappa^2\right]\bar{x}(\sigma,\zeta) = \frac{\epsilon}{\Delta\sigma}\int_0^{\zeta}d\zeta' F(\zeta')\int_{\sigma}^{\sigma+\Delta\sigma}d\sigma' w\left[\frac{\omega(\sigma')}{\omega_0}(\zeta-\zeta')\right]x(\sigma',\zeta'), \quad (3)$$

where

$$\bar{x}(\sigma,\zeta) \equiv \frac{1}{\Delta\sigma}\int_{\sigma}^{\sigma+\Delta\sigma}d\sigma' x(\sigma',\zeta). \quad (4)$$

We assume that the length $\Delta\sigma$ can be chosen small enough so that there is no appreciable beam breakup over $\Delta\sigma$. We can then replace $\bar{x}$ by x and take x out of the second integral in eq. (3). On the other hand, we assume $\Delta\sigma$ is large enough so that between σ and $\sigma+\Delta\sigma$ the deflecting-mode frequency takes all possible values so that it can accurately be represented by the probability density $\mathcal{f}(\omega)$. In that case, we have

$$\frac{1}{\Delta\sigma}\int_{\sigma}^{\sigma+\Delta\sigma}d\sigma' w\left[\frac{\omega(\sigma')}{\omega_0}\zeta\right] = \int_{-\infty}^{+\infty}d\omega\, w\left[\frac{\omega}{\omega_0}\zeta\right]\mathcal{f}(\omega) \equiv \hat{w}(\zeta), \quad (5)$$

where $\hat{w}(\zeta)$ is the "effective wake function." In Paper 1 we showed

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that, for a symmetric normalized probability density defined as $\tilde{g}(Z) \equiv \omega_0 \tilde{f}[\omega_0(Z+1)]$ where $Z \equiv (\omega/\omega_0)-1$, the effective wake function is

$$\hat{w}(\zeta) = 2\pi g(\zeta) w(\zeta) , \tag{6}$$

in which $\tilde{g}(Z)$ is the Fourier transform of $g(\zeta)$. If these assumptions are satisfied, then the equation of transverse motion takes the form

$$\left[\frac{d^2}{d\sigma^2} + \kappa^2\right] x(\sigma,\zeta) = \epsilon \int_0^{\zeta} d\zeta' \hat{w}(\zeta-\zeta') F(\zeta') x(\sigma,\zeta') . \tag{7}$$

In what follows, we solve eq. (7) for transient CBBU using the methods of Paper 1.

Gaussian Probability Density

According to eq. (6), the effective wake function corresponding to a Gaussian probability density of deflecting-mode frequencies with mean ω_0 and standard deviation $\Delta\omega$ can be written in the form

$$\hat{w}(\zeta) = w(\zeta)\exp(-\alpha\zeta^2) , \tag{8}$$

in which $\alpha \equiv (\Delta\omega)^2/2\omega_0^2$. The Fourier transform of the wake function is

$$\tilde{w}(Z) = \int_{-\infty}^{\infty} d\zeta e^{-iZ\zeta} \hat{w}(\zeta) = \frac{i}{4}\left[\frac{\pi}{\alpha}\right]^{1/2} [W(z_-) - W(z_+)] , \tag{9}$$

in which

$$W(z) \equiv e^{-z^2} erfc(-iz), \quad z_{\pm} \equiv \frac{\theta \pm 1}{2\sqrt{\alpha}}, \text{ and } \quad \theta \equiv i\left[\frac{1}{2Q} + iZ\right] .$$

We assume $|z_{\pm}| \gg 1$ because $\sqrt{\alpha} \ll 1$. Thus, we take [6]

$$W(z) \simeq \frac{i}{\sqrt{\pi}}\left[\frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5}\right], \tag{10}$$

$$\frac{dW}{dz} \simeq -\frac{i}{\sqrt{\pi}}\left[\frac{1}{z^2} + \frac{3}{2z^4}\right]. \tag{11}$$

Substituting for $z_{\pm}$ and keeping terms through order α gives

$$\hat{w}(\theta) \simeq \frac{1}{1-\theta^2} + 2\alpha\frac{1+3\theta^2}{(1-\theta^2)^3} , \tag{12}$$

$$\frac{d\hat{w}}{d\theta} \simeq \frac{2\theta}{(1-\theta^2)^2} + 24\alpha\theta\frac{1+\theta^2}{(1-\theta^2)^4} . \tag{13}$$

We consider a direct-current beam generating an impulse excitation of the wake field, i.e., the beam enters the linac such that $x(0,\zeta) = x_0\delta(\zeta)$. Results for bunched beams or for other initial conditions can be gleaned from this case by inspection of Paper 1. Solving the equation of transverse motion results in an expression for the transverse displacement $x(\sigma,\zeta)$ involving an integral over θ. An analytic expression for the transient CBBU growth rate Γ valid when BBU is pronounced is found by evaluating this integral by steepest descent. The saddle points are calculated by setting $f'(\theta)=0$, where

$$f(\theta) \equiv -i\zeta\left[\theta + \frac{s_2}{s_1}\sqrt{1-s_1\hat{w}(\theta)}\right] , \tag{14}$$

in which $s_1 = \epsilon/\kappa^2$ and $s_2 = \epsilon\sigma/\kappa\zeta$. In terms of the effective wake function, the equation for the saddle points θ_s is

$$[\hat{w}'(\theta_s)]^2 = \frac{4}{s_2^2}[1 - s_1\hat{w}(\theta_s)] . \tag{15}$$

Upon substituting from eqs. (12) and (13), retaining terms through order α, and letting $\psi \equiv \theta_s^2$, we are left with

$$\begin{aligned}[(\psi-1)^4 + s_1(\psi-1)^3 - s_2^2\psi](\psi-1)^2 \\ +2\alpha[s_1(3\psi+1)(\psi-1)^3 - 12s_2^2\psi(\psi+1)] = 0 .\end{aligned} \tag{16}$$

The growth rate is given by $\Gamma\zeta = Re[f(\theta_s)]$; thus, upon solving eq. (16) for ψ, we can evaluate Γ from

$$\Gamma = Im\left\{\sqrt{\psi}\left[1 + \frac{s_2^2/s_1}{(\psi-1)^2}\left[1 + 12\alpha\frac{\psi+1}{(\psi-1)^2}\right]\right]\right\} . \tag{17}$$

Because a distribution of deflecting-mode frequencies will have its greatest influence on long pulses, we restrict our discussion to long pulses (domains A and B of Paper 1) for which s_2 is small. For these cases, we write ψ in the form

$$\psi \simeq 1 + \Delta + \alpha\psi_1, \ |\Delta| \ll 1, \ \alpha\psi_1 \ll \Delta; \tag{18}$$

where Δ is given by

$$\Delta^4 + s_1\Delta^3 - s_2^2 \simeq 0 . \tag{19}$$

Upon substituting ψ from eq. (18) into eqs. (16) and (17) and retaining terms to order α, we find

$$\psi_1 \simeq \frac{8(6s_2^2 - s_1\Delta^3)}{\Delta^2(4\Delta^3 + 3s_1\Delta^2 - s_2^2)} , \tag{20}$$

$$\Gamma \simeq Im\left\{\frac{\Delta}{2}\left[1 + \frac{2s_2^2/s_1}{\Delta^3}\right] + \frac{\alpha}{2}\left[\left[1 - \frac{4s_2^2/s_1}{\Delta^3}\right]\psi_1 + \frac{48s_2^2/s_1}{\Delta^4}\right]\right\} . \tag{21}$$

Having established the needed formalism here and in Paper 1, we can now find the saddle points, growth rates, and beam envelopes by inspection. For each domain they are as follows:

Domain A: $s_2 \ll s_1^{1/2}$ and $s_2 \ll s_1^2$ (weak focusing, weak BBU coupling, long pulse length); $\Delta\omega/\omega_0 \ll (s_2^2/s_1)^{1/3}$. In terms of the dimensionless parameter

$$E_A \equiv \left[\frac{s_2^2}{s_1}\right]^{1/3} \zeta = (\epsilon\sigma^2\zeta)^{1/3} , \tag{22}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1 + \frac{E_A}{\zeta}e^{2\pi i/3} + \alpha\frac{40}{3}\frac{\zeta}{E_A}e^{-2\pi i/3}, \tag{23a}$$

$$\Gamma \simeq \frac{3\sqrt{3}}{4}\frac{E_A}{\zeta} - \alpha 2\sqrt{3}\frac{\zeta}{E_A}, \tag{23b}$$

and the beam envelope (with infinite Q) is given by

$$|x(\sigma,\zeta)| \simeq x_0 \frac{\sqrt{E_A}}{\zeta\sqrt{6\pi}} \exp\left[\frac{3\sqrt{3}}{4}E_A - \sqrt{3}\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\zeta^2}{E_A}\right]. \quad (24)$$

The exponent reaches its maximum value

$$(\Gamma\zeta)_{max} = \frac{3\sqrt{3}}{5}\left(\frac{3}{20}\right)^{1/4}\left(\frac{\omega_0}{\Delta\omega}\epsilon\sigma^2\right)^{1/2} \quad (25)$$

at

$$\zeta_{max} = \left(\frac{3}{20}\right)^{3/4}\left[\left(\frac{\omega_0}{\Delta\omega}\right)^3\epsilon\sigma^2\right]^{1/2} \quad (26)$$

These are the same results found by Gluckstern, Neri, and Cooper [5] for a beam of δ-function bunches in which case ζ is replaced by $M\omega_0\tau$, where M is the bunch number, and τ is the bunch period. These authors use a different method of calculation for which they claim to assume the bandwidth of the frequency distribution is small compared to the intrinsic bandwidth of the deflecting mode associated with finite Q [4,5]. However, they have confirmed the validity of eq. (24) with numerical simulations in the absence of this assumption. With our method we make no assumption about the bandwidth of the deflecting mode and have included the case of infinite Q at the outset.

Domain B: $s_1^2 \ll s_2 \ll 1$ (strong focusing, moderate BBU coupling, moderate pulse length); $\Delta\omega/\omega_0 \ll \sqrt{s_2}$. In terms of the dimensionless parameter

$$E_B \equiv \sqrt{s_2}\zeta = (\epsilon\sigma\zeta/\kappa)^{1/2}, \quad (27)$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1 + i\frac{E_B}{\zeta}\left(1 + i\frac{s_1\zeta}{4E_B}\right) - 12i\alpha\frac{\zeta}{E_B}\left(1 - \frac{s_1\zeta}{3E_B}\right), \quad (28a)$$

$$\Gamma \simeq \frac{E_B}{\zeta} - 4\alpha\frac{\zeta}{E_B}, \quad (28b)$$

and the beam envelope (for infinite Q) is given by

$$|x(\sigma,\zeta)| \simeq x_0\frac{\sqrt{E_B}}{2\zeta\sqrt{2\pi}}\exp\left[E_B - 2\left(\frac{\Delta\omega}{\omega_0}\right)^2\frac{\zeta^2}{E_B}\right]. \quad (29)$$

The exponent reaches its maximum value

$$(\Gamma\zeta)_{max} = \frac{2}{3\sqrt{6}}\frac{\omega_0}{\Delta\omega}\frac{\epsilon\sigma}{\kappa} \quad (30)$$

at

$$\zeta_{max} = \frac{1}{6}\left(\frac{\omega_0}{\Delta\omega}\right)^2\frac{\epsilon\sigma}{\kappa}. \quad (31)$$

Lorentzian Probability Density

In Paper 1 we show that a Lorentzian probability density of deflecting-mode frequencies with mean ω_0 and half-width at half maximum $\Delta\omega$ generates an effective wake function with infinite Q which is identical to the single-mode wake function with finite $Q=\omega_0/2\Delta\omega$. Thus, the beam envelopes found in Paper 1 apply directly, and for impulse excitation of the wake field, they are:

Domain A:

$$|x(\sigma,\zeta)| \simeq x_0\frac{\sqrt{E_A}}{\zeta\sqrt{6\pi}}\exp\left[\frac{3\sqrt{3}}{4}E_A - \frac{\Delta\omega}{\omega_0}\zeta\right]. \quad (32)$$

The exponent reaches its maximum value

$$(\Gamma\zeta)_{max} = \frac{3^{3/4}}{4}\left(\frac{\omega_0}{\Delta\omega}\epsilon\sigma^2\right)^{1/2} \quad (33)$$

at

$$\zeta_{max} = \frac{3^{3/4}}{8}\left[\left(\frac{\omega_0}{\Delta\omega}\right)^3\epsilon\sigma^2\right]^{1/2}. \quad (34)$$

Domain B:

$$|x(\sigma,\zeta)| \simeq x_0\frac{\sqrt{E_B}}{2\zeta\sqrt{2\pi}}\exp\left[E_B - \frac{\Delta\omega}{\omega_0}\zeta\right]. \quad (35)$$

The exponent reaches its maximum value

$$(\Gamma\zeta)_{max} = \frac{1}{4}\frac{\omega_0}{\Delta\omega}\frac{\epsilon\sigma}{\kappa} \quad (36)$$

at

$$\zeta_{max} = \frac{1}{4}\left(\frac{\omega_0}{\Delta\omega}\right)^2\frac{\epsilon\sigma}{\kappa}. \quad (37)$$

Conclusion

We have calculated the transient cumulative beam breakup with Gaussian and Lorentzian probability densities of deflecting-mode frequencies. The results for long pulses in the limits of weak and strong focusing indicate that the Gaussian generates the fastest decay to the steady state. Specifically, during decay with weak focusing the exponent behaves like $\Gamma\zeta \propto -\zeta^{5/3}$ for the Gaussian and like $\Gamma\zeta \propto -\zeta$ for the Lorentzian, and with strong focusing the exponent behaves like $\Gamma\zeta \propto -\zeta^{3/2}$ for the Gaussian and again like $\Gamma\zeta \propto -\zeta$ for the Lorentzian. This is caused by the faster decay of the effective wake function associated with the Gaussian, and confirms our conjecture that the rate of approach to the steady state depends strongly on the shape of the probability density. The mathematical expressions for the locations and magnitudes of the beam-envelope peaks are similar. The physical meaning of the "bandwidth" $\Delta\omega$ is different in the two cases, however. For the Gaussian it is the standard deviation, and for the Lorentzian it is the half-width at half maximum.

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