TRANSVERSE PHASE-SPACE EVOLUTION IN NON-STATIONARY CHARGED PARTICLE BEAMS*

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Abstract

The transverse dynamics of a nonrelativistic, mismatched, one-dimensional sheet beam propagating through a continuous, linear focusing channel is investigated. The investigation is based on the Fokker-Planck equation in which the relaxation rate and diffusion coefficient are calculated from a simple model of turbulence resulting from charge redistribution.

Introduction

This paper concerns the dynamics of transverse emittance growth for nonrelativistic space-charge-dominated beams in continuous, linear focusing channels. Mismatches in both the density profile and root-mean-square (rms) beam size [1,2] contribute to the emittance growth. A mismatched beam carries excess total energy which can be thermalized if nonlinear forces, instabilities, and/or collisions are present. The resultant heating generates emittance growth, and the magnitude of the possible emittance growth can be calculated from the excess total energy [2].

The earliest evolutionary stage after the beam enters the focusing channel is marked by charge redistribution. In a zero-temperature beam, the trajectories of the individual beam particles do not cross, and laminar flow is present. However, if the initial density profile gradually falls to zero at the beam edge, then laminar flow will cease very quickly, at about one-quarter of a plasma period after injection, at which time particle trajectories originating in that part of the beam with the lowest initial density will cross [3]. The termination of laminar flow is marked by nonlinearity in the form of discontinuous shock-like behavior associated with wave-breaking in phase space and the onset of irreversible dynamics [3,4]. The charge-redistribution phase in a warm beam terminates similarly [3].

While the charge-redistribution phase lends itself to simple analysis, the subsequent evolution is more complicated. Yet, an understanding of this evolution is essential because it will dominate during almost the entire transport of real beams. To date, this problem has been investigated by way of both direct experimentation [5,6] and computational N-body simulations [1,7,8]. These investigations have provided an accurate picture of the dynamics for the special cases considered, but they have not provided a synthesized account of the underlying physics. The purpose of the present paper is to integrate the essential physics into a single formalism reproducing the salient dynamical features revealed in prior investigations.

Formulation of the Problem

During the charge-redistribution phase, the beam evolves toward a density profile which is nearly uniform, particularly if the beam is strongly dominated by space charge. If this nearly uniform beam is mismatched to the transport channel, it carries free energy available for thermalization. The shock-like behavior and wave breaking in phase space ending this phase are nonlinear phenomena which may trigger strong, localized turbulent fluctuations. Resonant coupling between beam particles and the turbulence provides a mechanism for converting free energy, now contained in the turbulent fluctuations, into heat. In strong turbulence the heating will occur very rapidly, on a time scale of the order of a plasma period [9]. Because thermalization occurs at the expense of the energy contained in the turbulent fluctuations, the turbulence also weakens on the same time scale.

Beam particles slow down by colliding with turbulent fluctuations of the mean electric field. This anomalous resistivity also occurs on a time scale of the order of a plasma period in strong turbulence. The associated average collision frequency is then $\sim g^{-1} \equiv n\lambda_D^3$ times larger than in a quiescent plasma [9], where n is the number density and λ_D is the Debye length. In space-charge-dominated beams g^{-1} is large and collective effects appear. In weak turbulence the average collision frequency is $\sim g^{1/2}$ times smaller than in strong turbulence.

Scattering off turbulent fluctuations is considered to impart white noise on particle trajectories, thereby establishing a Markov process resulting in Brownian motion. The scattering creates dynamical friction and diffusion, causing relaxation to occur on a time scale which evolves from short to long as the turbulence dissipates. Heating and relaxation of the beam eject a fraction of the particles into large-amplitude orbits causing a halo to form with associated emittance growth. Because the relaxation time can be very short, these processes can occur during beam transport.

These considerations motivate using the Fokker-Planck equation for the evolution of a coarse-grained distribution function in the phase space of a single beam particle. We let $W(\vec{x},\vec{u},t;\vec{x}_0,\vec{u}_0)d\vec{x}d\vec{u}$ denote the probability of finding a particle with position $\vec{x}$ and velocity $\vec{u}$ in the range $(\vec{x},\vec{x}+d\vec{x})$, $(\vec{u},\vec{u}+d\vec{u})$, respectively, at time t given it started at $(\vec{x}_0,\vec{u}_0)$ at t=0. The Fokker-Planck equation is

$$\frac{\partial W}{\partial t} + \vec{u}\cdot\vec{\nabla}_x W + \vec{K}\cdot\vec{\nabla}_u W = \beta\vec{\nabla}_u\cdot(W\vec{u}) + \beta\frac{kT}{m}\nabla_u^2 W, \qquad (1)$$

where $\vec{K}$ is the net force per unit mass m, β is the relaxation rate, k is Boltzmann's constant, and T is the temperature. The net force is the superposition of the focusing force and the space-charge force found from the potentials Φ_f and Φ_s, respectively, and thus

$$\vec{K} = -\frac{q}{m}\vec{\nabla}_x(\Phi_f + \Phi_s)\,, \qquad (2)$$

where q is the particle charge. According to Poisson's equation, Φ_s is given by

$$\nabla_x^2\Phi_s(\vec{x},t) = -\frac{\mathbf{N}q}{\epsilon_o}\int d\vec{u}\int d\vec{u}_0\int d\vec{x}_0\, W(\vec{x},\vec{u},t;\vec{x}_0,\vec{u}_0)\,W(\vec{x}_0,\vec{u}_0)\,, \qquad (3)$$

where $\mathbf{N}$ is the total line density, ϵ_o is the permittivity of free space, and $W(\vec{x}_0,\vec{u}_0)$ is the single-particle distribution function at t=0.

The temperature will generally depend both on position and time. However, if the turbulence is initially strong enough to induce rapid heating and relaxation, and if most of the beam particles scatter off the strong fluctuations, then to a reasonable approximation the beam may be regarded as isothermal with an increasing temperature which saturates as the turbulence weakens. Letting β_s denote the relaxation rate in the presence of strong turbulence, we adopt an exponential

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model of turbulent heating:

$$T(t) = T_\infty + (T_0 - T_\infty)\exp(-\beta_s t)\,. \quad (4)$$

Starting from temperature T_0, the beam strives to reach a Maxwell-Boltzmann distribution with temperature T_∞.

To obtain a self-consistent solution, one must solve eqs. (1)-(3) simultaneously using eq. (4) to represent turbulent heating. For mathematical simplicity we treat a one-dimensional (1D) sheet beam centered on the focusing-channel axis with a focusing force $-m\omega^2 x$. We pick a particular beam cross-section and calculate its properties as it moves down the transport channel. Our treatment is easily generalized to higher dimensions, and the qualitative features of the results should be insensitive to the dimensionality.

Analytic Solution with Harmonic-Oscillator Orbits

Assuming $\beta \ll \beta_s$ to be a constant in eq. (1) associated with weak residual turbulence, and using a harmonic-oscillator model of the particle orbits, we can solve the problem in closed form with standard methods [10]. We take $-m\omega_n^2 x$ to be the net restoring force, where ω_n is the particles' orbital frequency. This assumed force replaces Poisson's equation, and thus self-consistency is sacrificed. However, the approach is instructive because it leads to simple results exhibiting many of the prominent features of self-consistent solutions discussed below. Moreover, the resulting distribution function may be used to calculate any of the moments of x and u in terms of elementary functions.

In the limit of instantaneous heating ($\beta_s \to \infty$), the rms beam size $a \equiv \langle x^2 \rangle^{1/2}$ and emittance $\epsilon \equiv [\langle x^2\rangle \langle u^2\rangle - \langle xu\rangle^2]^{1/2}$, normalized to their values at t=0, are respectively found to be

$$\frac{a^2(t)}{a_0^2} = e^{-\beta t}\left(\frac{\beta}{2\omega_1}\sin\omega_1 t + \cos\omega_1 t\right)^2 + \frac{a_\infty^2}{a_0^2} \times \left\{1 + e^{-\beta t}\left[\frac{\omega_n^2}{\omega_1^2}\left(\frac{\beta^2}{4\omega_n^2}\cos 2\omega_1 t - 1 + \frac{T_0}{T_\infty}\sin^2\omega_1 t\right) - \frac{\beta}{2\omega_1}\sin 2\omega_1 t\right]\right\}, \quad (5)$$

$$\frac{\epsilon^2(t)}{\epsilon_0^2} = \left(1 - \frac{T_\infty}{T_0}\right)e^{-2\beta t} - \frac{T_\infty}{T_0}e^{-\beta t}\left[\frac{\omega_n^2}{\omega_1^2}\left(\frac{\beta^2}{4\omega_n^2}\cos 2\omega_1 t - 1\right) - \frac{\beta}{2\omega_1}\sin 2\omega_1 t\right] + \frac{\epsilon_\infty^2}{\epsilon_0^2}\left\{1 + \left(1 - \frac{T_0}{T_\infty}\right)e^{-2\beta t} + e^{-\beta t} \times \left[\left(2 - \frac{T_0}{T_\infty}\right)\frac{\omega_n^2}{\omega_1^2}\left(\frac{\beta^2}{4\omega_n^2}\cos 2\omega_1 t - 1\right) - \frac{T_0}{T_\infty}\frac{\beta}{2\omega_1}\sin 2\omega_1 t\right]\right\}, \quad (6)$$

where $\omega_1^2 \equiv \omega_n^2 - \beta^2/4$, $T_\infty / T_0 = (a_0/a_\infty)^2(\epsilon_\infty/\epsilon_0)^2$, and a_0, ϵ_0, a_∞, ϵ_∞ are values at $t=0$, $t\to\infty$, respectively. The ratios a_∞/a_0 and $\epsilon_\infty/\epsilon_0$ can be determined from force-balance and energy-conservation arguments [2]. Example plots of a and ϵ appear in Fig. 1(a). Using a constant relaxation rate β results in the rise time of the emittance being determined by β, even with instantaneous heating from the initially strong turbulence.

Self-Consistent Numerical Solution

We solve eqs. (1)-(3) self-consistently by decomposing the single-particle distribution function into complete sets of orthonormal polynomials [11]. For the 1D sheet beam a natural choice for both coordinate and velocity space is the set of 1D quantum mechanical harmonic-oscillator eigenfunctions:

$$W(x,u,t) = \phi_0\psi_0 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n^m \phi_m \psi_n\,, \quad (7)$$

$$\phi_j(x,t) = \left[\frac{1}{2^j j!}\left(\frac{A}{\pi}\right)^{1/2}\right]^{1/2} H_j(\sqrt{A}\,x)\,e^{-Ax^2/2}\,, \quad A \equiv \frac{m\omega_0^2}{2kT(t)}\,; \quad (8)$$

$$\psi_k(u,t) = \left[\frac{1}{2^k k!}\left(\frac{\alpha}{\pi}\right)^{1/2}\right]^{1/2} H_k(\sqrt{\alpha}\,u)\,e^{-\alpha u^2/2}\,, \quad \alpha \equiv \frac{m}{2kT(t)}\,; \quad (9)$$

where $c_n^m(t)$ are time-dependent coefficients, ω_0 is a reference frequency, and H_i are Hermite polynomials. By symmetry, the coefficients c_n^m for which $m+n$ is odd are zero. Moments of special interest are the number density and rms beam size and emittance, which are respectively given by

$$n(x,t) = \mathbf{N}\phi_0 \sum_{p=0}^{\infty} c_0^{2p}\phi_{2p}\,, \quad (10)$$

$$\frac{a(t)}{a(0)} = \left\{\frac{T(t)}{T_0}[1 + \sqrt{2}\,c_0^2(t)]\right\}^{1/2}, \quad (11)$$

$$\frac{\epsilon(t)}{\epsilon(0)} = \frac{T(t)}{T_0}\left\{[1 + \sqrt{2}\,c_0^2(t)][1 + \sqrt{2}\,c_2^0(t)] - [c_1^1(t)]^2\right\}^{1/2}. \quad (12)$$

Integrating $n(x,t)$ over all values of x must always yield $\mathbf{N}$; consequently, we must have $c_0^0(t) = 1$ for all t.

If turbulence is initially strong enough to cause rapid heating and relaxation, and if most particles scatter significantly off the electric-field fluctuations as they orbit, then to a reasonable approximation the relaxation rate may be considered a function of time only. The turbulence is expected to pass rapidly from strong to weak, so we use eq. (4) for turbulent heating and, as before, we let $\beta \ll \beta_s$ be a constant in eq. (1) to represent persistent weak turbulence.

Upon substituting for $W(x,u,t)$ in eqs. (1)-(3) and using the orthonormality of ϕ_j and ψ_k along with the recurrence relations for H_i, we obtain the following infinite system of coupled differential equations for the coefficients c_n^m:

$$\dot{c}_n^m = -(\dot{T}/2T)\left[\sqrt{m(m-1)}\,c_n^{m-2} + \sqrt{n(n-1)}\,c_{n-2}^m + (m+n)c_n^m\right] - n(\beta/\omega_0)c_n^m + \sqrt{m}\left(\sqrt{n}\,c_{n-1}^{m-1} + \sqrt{n+1}\,c_{n+1}^{m-1}\right) - (\omega/\omega_0)^2\sqrt{n} \times \left(\sqrt{m}\,c_{n-1}^{m-1} + \sqrt{m+1}\,c_{n-1}^{m+1}\right) + (\omega_p/\omega_0)^2\sqrt{T_0/T}\sqrt{2n}\sum_{j=0}^{\infty} c_{n-1}^j \kappa_m^j\,, \quad (13)$$

where the dot denotes differentiation with respect to $\zeta \equiv \omega_0 t$;

$$\omega_p^2 = \frac{\mathbf{N}q^2\omega_0}{2\epsilon_o\sqrt{\pi m k T_0}} \quad (14)$$

is the plasma frequency; and

$$\kappa_p^q = \mu_p^q - \sum_{n=1}^{\infty} c_0^{2n}\nu_p^q(n)\,, \quad (15)$$

in which

$$\mu_p^q = \frac{(-1)^{(p-q-1)/2}(p+q+1)!}{2^{p+q+1/2}\sqrt{p!\,q!}\,(p-q)[(p+q+1)/2]!}\,, \quad (16)$$

$$\nu_p^q(n) = \frac{\Gamma[n+(q-p)/2]\,\Gamma[1-n+(q+p)/2]\,\Gamma[n-(q-p)/2]}{\pi\sqrt{2\pi p!\,q!}\,(2n)!}\,. \quad (17)$$

This infinite system of coupled differential equations is equivalent to the coupled Fokker-Planck and Poisson equations. The second-to-last term in eq. (13) originates from the linear focusing force, and the last term originates from the nonlinear space-charge force.

Upon truncating the system at $m=M$ and $n=N$, it can be solved by standard numerical integration [12]. We set $M=4$, $N=3$ to allow for two time-dependent coefficients c_0^2, c_0^4 in eq. (10) for the density. This is sufficient to represent the underlying physics and establish a first approximation to the numerical solution; however, the accuracy can be improved as desired by continuously increasing M, N until essentially no change is seen in the results. Our truncation leaves nine equations with nine unknown coefficients. They are solved for a beam which is initially Maxwellian in velocity space with a Gaussian density profile of standard deviation $\sigma_o^2 = 1/2A(0)$; the initial conditions are $c_n^m(\zeta=0) = \delta_{mn}$.

Example plots of a and ϵ in a self-consistent calculation appear in Fig. 1(b). Though the details of these curves differ from those of the analytic solution [Fig. 1(a)], the qualitative features are similar. In particular the relaxation rate β, assumed here to be constant, determines the rise time of the emittance. Plots of the corresponding density profile are depicted in Fig. 2. As the beam relaxes, more and more particles are injected into high-amplitude orbits.

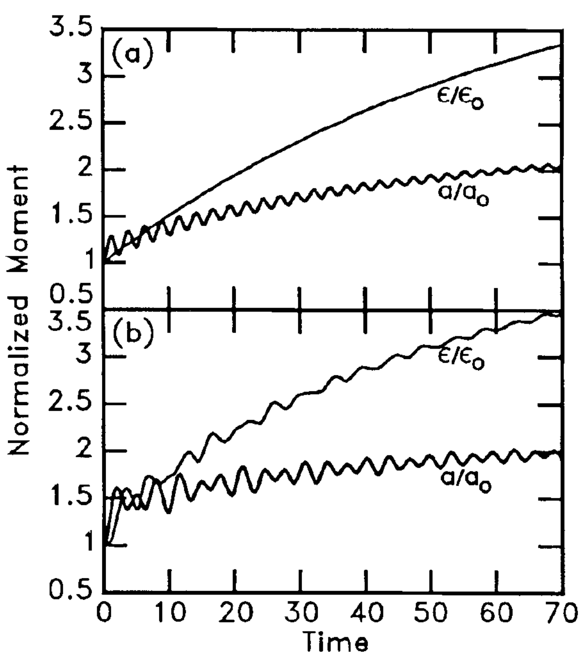

Fig. 1. RMS beam size and emittance, normalized to $t=0$, (a) from harmonic-oscillator model of particle orbits with $(\omega_n/\omega_0)^2=1.5$ and $\beta=0.015$ in the limit $\beta_s \to \infty$; (b) calculated self-consistently with $(\omega/\omega_0)^2=1.5$, $(\omega_p/\omega_0)^2=1.4$, $\beta_s/\omega_0=1.0$, and $\beta/\omega_0=0.015$.

Discussion

As described here, the mismatched beam evolves on time scales short compared to the transit time of the beam through the transport channel. The associated spread in the density profile may be problematic if the transport elements have small bore-hole apertures. For example, in the design of high-current machines for long-term, continuous-wave operation, radioactivation from beam impingement is a concern. Since the growth in beam size and emittance correlates to the degree of mismatch [2], the cure involves both reducing sources of mismatch and increasing the bore-hole sizes.

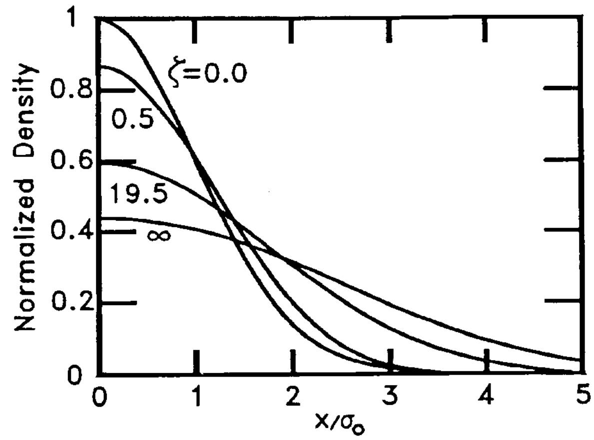

Fig. 2. Snapshots of density found with parameters of Fig. 1(b).

We based our calculations on a simple model of turbulence which was assumed to be strong initially, causing rapid heating of the beam, and to dissipate quickly to persistent weak turbulence. The relaxation rate was taken to be a constant associated with weak turbulence, and was regarded as a free parameter. Since both the diffusion coefficient and relaxation rate in the Fokker-Planck equation are determined by the spectrum of electric-field fluctuations, it would be interesting to study the time-dependence of the fluctuation spectrum using N-body simulations. This may lead to more accurate models of the Fokker-Planck coefficients which could then be readily incorporated into our self-consistent numerical formulation.

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