EMITTANCE GROWTH IN HEAVY ION RECIRCULATORS*

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Abstract

We make analytic estimates of emittance growth in recirculating heavy ion fusion beams by developing a set of moment equations which include the first order effects of energy dispersion of particles traversing a circular or racetrack-shaped accelerator, and by assuming transverse energy conservation. We compare our results to those of detailed 3-D simulations of small scale recirculators using the WARP code (ref. [1]). We further apply our results to the recirculator driver designs of ref. [2].

Introduction

In order to focus a heavy ion beam onto a sufficiently small spot at a target in an inertial confinement fusion reactor, the normalized emittance of the beam cannot be too large. The growth of the normalized emittance of an accelerated beam is also of interest for many other applications in which high brightness is required. The concept of transverse energy conservation has been used before in the study of emittance growth in particle beams. Emittance growth associated with non-uniform space-charge distributions has been studied in, for example, refs. [4]- [6]. Emittance growth due to initial beam displacements and mismatches with and without space-charge has been studied in, refs. [7]-[9], and references therein. These studies were generally concerned with the emittance growth in straight focusing channels. In the induction recirculator proposed in ref. [2], the beams propagate in a FODO focusing channel, with phase advances that are highly depressed due to space charge. In addition, bends are present, which provide a displacement in the center of oscillation for ions which are off of the design energy.

In this paper, we estimate the growth from a single transition from bend to straight including the effects of both FODO focusing and space charge, as well as energy dispersion in the bends. (See also ref. [8] for an estimate of emittance growth due to the transitions in the absence of space charge.) On a transition from a bend to a straight section, or from a straight section to a bend, if the transition is sufficiently sharp, the beam becomes mismatched. We assume that a small non-linear force acts to phase mix particles, and we find the asymptotic emittance of such a beam. Further, if we assume that the process of phase mixing is completed before the beam goes through another straight/bend transition, we may calculate the emittance growth through a "racetrack" configuration consisting of two 180° bends and two straight sections, even without a detailed knowledge of the rate at which the phase mixing occurs.

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Model Equations of Motion

We assume that the equations of motion of for each ion are given by,

$$x_i'' = -k_{\beta 0}^2(x_i - x_{mi}) + k_{sx}^2(x_i - x_c) - \frac{\partial h_{nl}(x_i, y_i)}{\partial x}. \quad (1)$$

$$y_i'' = -k_{\beta 0}^2 y_i + k_{sy}^2(y_i - y_c) - \frac{\partial h_{nl}(x_i, y_i)}{\partial y}. \quad (2)$$

Here, x_i and y_i are the coordinates of the ith particle in a particular slice in z, which is traveling in the $+z$ direction, and where $'$ indicates derivative with respect to z; $k_{\beta 0} \cong \sigma_0/(2L)$ represents FODO focusing in the smooth approximation, where σ_0 is the undepressed phase advance, and L is the lattice half-period; $x_{mi} = \eta(\delta p_i/p) \cong (1/k_{\beta 0}^2\rho)(\delta p_i/p)$, where η is the "η-function" representing the displacement from the design orbit of a particle with unit fractional momentum error, $\delta p_i/p$ is the fractional difference between the momentum of the ith particle and the design momentum, and ρ is the average radius of curvature due to bending magnets; $K \equiv 2qI/(\beta^3 A I_o)$ is the perveance, where q is the charge state of the ions, A is the atomic mass of the ions, β is the velocity of the ions in units of c, $I_o \equiv m_p c^3/e$ is the proton Alfven current ($\cong$31 MA). h_{nl} is an unspecified non-linear potential that is a function of x_i and y_i.

Note that throughout this paper Δ is reserved for the two argument operator satisfying: $\Delta ab = \langle ab \rangle - \langle a \rangle\langle b \rangle$ (e.g. $\Delta x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$), where $\langle\rangle$ indicates average over all particles in a slice; $x_c \equiv \langle x \rangle$, and $y_c \equiv \langle y \rangle$. Also,

$$k_{sx}^2 \equiv \frac{K}{2(\Delta x^2 + (\Delta x^2 \Delta y^2)^{1/2})};$$

$$k_{sy}^2 \equiv \frac{K}{2(\Delta y^2 + (\Delta x^2 \Delta y^2)^{1/2})}; \quad (3)$$

Eqs. (1) and (2) represent in an approximate way, the effects of: focusing, space charge, dispersion in a bend, and external non-linearities in the focusing field. The physical approximations that have been made include the following: (1) Focusing is smooth and not a function of z ($k_{\beta_0} = \sigma_0/(2L)$ is constant). (2) Eqs. (1) and (2) have been linearized in the small quantities $k_{\beta 0}x_i$, $k_{\beta 0}y_i$, and $\delta p_i/p$. (The non-linear term h_{nl} has also been included in some of the derivations). (3) The non-linearity is small: ($|h_{nl}| << |k_{\beta 0}^2 x_i^2|, |k_{\beta 0}^2 y_i^2|$). (Terms which are non-linear in $\delta p_i/p$, such as $k_{\beta 0}x_i\delta p_i/p$, have been ignored completely.) (4) Space charge forces depend only on lowest order moments. (We have used the KV formula for the electrostatic potential, which is equivalent to assuming uniform density elliptical beam. Centroid position and semi-major axes are, however, allowed to vary with z). (5) Coasting beam: (p, β, and δp_i are constants). (6) The beam is non-relativistic: ($\beta << 1$).

Let $f(x, x', y, y', \frac{\delta p}{p}, z) = \mathrm{d}N/\mathrm{d}x\mathrm{d}x'\mathrm{d}y\mathrm{d}y'\mathrm{d}\frac{\delta p}{p}$ where $\mathrm{d}N$ is the number of particles within incremental phase volume $\mathrm{d}x\mathrm{d}x'\mathrm{d}y\mathrm{d}y'\,\mathrm{d}\frac{\delta p}{p}$.

Combining the model equations of motion with the Liouville equation yields the Vlasov equation for this problem:

$$\frac{\partial f}{\partial z} + x'\frac{\partial f}{\partial x} + \left(-k_{\beta 0}^2(x_i - x_{mi}) + k_{sx}^2(x_i - x_c) - \frac{\partial h_{nl}}{\partial x}\right)\frac{\partial f}{\partial x'} +$$

$$+y'\frac{\partial f}{\partial y} + \left(-k_{\beta 0}^2 y_i + k_{sy}^2(y_i - y_c) - \frac{\partial h_{nl}}{\partial y}\right)\frac{\partial f}{\partial y'} = 0. \quad (4)$$

The average of a variable a over the continuous distribution is given by:

$< a > (z) \equiv \int\int\int\int\int a f(x, x', y, y', \frac{\delta p}{p}, z)\mathrm{d}x\mathrm{d}x'\mathrm{d}y\mathrm{d}y'\mathrm{d}\frac{\delta p}{p}$.

Following ref. [3], we may multiply eq. (4) by linear or second order quantities and integrate over the distribution function, resulting in a set of equations for the first and second order moments:

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta x^2 = 2\Delta x x' \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta x'^2 = (-2k_{\beta 0}^2 + 2k_{sx}^2)\Delta x x' + 2k_{\beta 0}^2 \Delta x' x_m - 2\Delta(x'\tfrac{\partial h_{nl}}{\partial x}) \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta x x' = \Delta x'^2 - k_{\beta 0}^2\Delta x^2 + k_{sx}^2\Delta x^2 + k_{\beta 0}^2 \Delta x x_m - \Delta(x\tfrac{\partial h_{nl}}{\partial x}) \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta y^2 = 2\Delta y y' \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta y'^2 = (-2k_{\beta 0}^2 + 2k_{sy}^2)\Delta y y' - 2\Delta(y'\tfrac{\partial h_{nl}}{\partial y}) \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta y y' = \Delta y'^2 - k_{\beta 0}^2\Delta y^2 + k_{sy}^2\Delta y^2 - \Delta(y\tfrac{\partial h_{nl}}{\partial y}) \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta x x_m = \Delta x' x_m \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}\Delta x' x_m = -k_{\beta 0}^2\Delta x x_m + k_{sx}^2\Delta x x_m + k_{\beta 0}^2\Delta x_m^2 - \Delta(x_m\tfrac{\partial h_{nl}}{\partial x}) \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}x_c = x'_c \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}x'_c = -k_{\beta 0}^2 x_c + k_{\beta 0}^2 < x_m > - < \tfrac{\partial h_{nl}}{\partial x} > \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}y_c = y'_c \\
&\tfrac{\mathrm{d}}{\mathrm{d}z}y'_c = -k_{\beta 0}^2 y_c - < \tfrac{\partial h_{nl}}{\partial y} > \qquad (5)
\end{aligned}$$

Note that if $h_{nl} = 0$, eqs. (5) form 3 closed sets of (8,2,2) equations. If $h_{nl} \neq 0$, eqs. (5) form the beginning of an infinite set of equations.

Transverse Energy Conservation

We may define a transverse energy H:

$$\begin{aligned}
2H = &k_{\beta 0}^2(\Delta x^2 + \Delta y^2) + \Delta x'^2 + \Delta y'^2 - 2k_{\beta 0}^2\Delta x x_m \\
&-K\ln\left((\Delta x^2)^{1/2} + (\Delta y^2)^{1/2}\right) + 2 < h_{nl} > \\
&+k_{\beta 0}^2 x_c^2 + k_{\beta 0}^2 y_c^2 + x_c'^2 + y_c'^2
\end{aligned}$$

Use of eqs. (5) shows that:

$$\tfrac{\mathrm{d}}{\mathrm{d}z}H = \tfrac{\mathrm{d}}{\mathrm{d}z} < h_{nl} > \qquad (6)$$

Thus if h_{nl} is not a function of z, H is an invariant.

Emittance Growth

We define separate x and y emittances:

$$\epsilon_x^2 = 16(\Delta x^2\Delta x'^2 - \Delta x x'^2)$$
$$\epsilon_y^2 = 16(\Delta y^2\Delta y'^2 - \Delta y y'^2)$$

Again using eqs. (5) yields:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}z}\epsilon_x^2 = &32k_{\beta 0}^2(\Delta x^2\Delta x' x_m - \Delta x x'\Delta x x_m) \\
&+32\left(\Delta(x\tfrac{\partial h_{nl}}{\partial x})\Delta x x' - \Delta x^2\Delta(x'\tfrac{\partial h_{nl}}{\partial x})\right) \\
\tfrac{\mathrm{d}}{\mathrm{d}z}\epsilon_y^2 = &32\left(\Delta(y\tfrac{\partial h_{nl}}{\partial y})\Delta y y' - \Delta y^2\Delta(y'\tfrac{\partial h_{nl}}{\partial y})\right) \qquad (7)
\end{aligned}$$

Thus the emittance would be constant if non-linearities were not present ($h_{nl} = 0$) and the energy spread were absent ($x_m = 0$ for all particles.)

Equilibrium Beam

By setting eqs. (5) to zero, we may arrive at a set of conditions for which the moments remain constant:

$$\begin{aligned}
&\Delta x'^2 = (k_{\beta 0}^2 - k_{sx}^2)\Delta x^2 - k_{\beta 0}^2\Delta x x_m + \Delta(x\tfrac{\partial h_{nl}}{\partial x}) \\
&\Delta y'^2 = (k_{\beta 0}^2 - k_{sy}^2)\Delta y^2 + \Delta(y\tfrac{\partial h_{nl}}{\partial y}) \\
&\Delta x x_m = \tfrac{k_{\beta 0}^2\Delta x_m^2 - \Delta(x_m\frac{\partial h_{nl}}{\partial x})}{k_{\beta 0}^2 - k_{sx}^2} \\
&x_c = < x_m > - \tfrac{1}{k_{\beta 0}^2} < \tfrac{\partial h_{nl}}{\partial x} > \\
&y_c = -\tfrac{1}{k_{\beta 0}^2} < \tfrac{\partial h_{nl}}{\partial x} > \\
&\Delta x x' = \Delta y y' = \Delta x' x_m = \Delta(x'\tfrac{\partial h_{nl}}{\partial x}) = \Delta(y'\tfrac{\partial h_{nl}}{\partial y}) = 0 \\
&x'_c = y'_c = 0 \qquad (8)
\end{aligned}$$

Assuming that $h_{nl} = x_c = y_c = 0$ the equilibrium transverse energy can be written:

$$\begin{aligned}
2H_{eq} = &(2k_{\beta 0}^2 - k_{sx}^2)\Delta x^2 + (2k_{\beta 0}^2 - k_{sy}^2)\Delta y^2 \\
&-\tfrac{3k_{\beta 0}^4\Delta x_m^2}{k_{\beta 0}^2 - k_{sx}^2} - K\ln\left((\Delta x^2)^{1/2} + (\Delta y^2)^{1/2}\right) \qquad (9)
\end{aligned}$$

Note that for a given H_{eq}, the ratio of Δx^2 to Δy^2 is still unspecified. A further assumption is required to specify the final state of the beam. Hence, we assume that transverse energy equipartition results in a beam in which the two transverse temperatures are equal, i.e. $\Delta x'^2 = \Delta y'^2$. (Note, that we have implicitly assumed that the timescale for complete equipartition $[\Delta x'^2 = \Delta y'^2 = \Delta(\delta p/p)^2]$ is much larger than timescales of interest.) In the limit, that $\Delta x_m^2 << \Delta x^2$, the condition $\Delta x'^2 = \Delta y'^2$ yields the relation,

$$\Delta y^2 \cong \Delta x^2 - k_{\beta_0}^4\Delta x_m^2/k^4, \qquad (10)$$

where $k^2 \equiv k_{\beta 0}^2 - K/(4\Delta x^2)$.

Perpetual Bends

For a beam which is in equilibrium in a straight section, and then enters a bend, the beam becomes mismatched for the bend. Physically, particles that are not on the design momentum for the bend initially become spatially separated, creating non-linear space-charge forces, allowing phase mixing of the coherent mismatch oscillations, until a new equilibrium is reached. Initially, $\Delta y_0^2 = \Delta x_0^2$, and $\Delta x_0'^2 = \Delta y_0'^2 = k_0^2\Delta x_0^2$, where subscript 0 indicates initial value, and all other moments are equal to zero. The initial transverse energy satisfies $2H_0 = 2(k_{\beta 0}^2 + 2k^2)\Delta x_0^2 - K\ln[2(\Delta x_0^2)^{1/2}]$. Assuming that the final transverse energy equals the initial yields the following change in emittance squared in an abrupt transition from straight to bend:

$$\epsilon_x^2 - \epsilon_{x0}^2 \cong 4\tfrac{k_{\beta 0}^4}{k^4}(7k_{\beta 0}^2 + 5k^2)\Delta x_0^2\Delta x_m^2 \qquad (11)$$

$$\epsilon_y^2 - \epsilon_{y0}^2 \cong 4\frac{k_{\beta_0}^4}{k^4}(5k_{\beta 0}^2 + 3k^2)\Delta x_0^2 \Delta x_m^2 \tag{12}$$

Racetracks

In a recirculator, that is composed of two 180° bends connected by two straight sections in the shape of a racetrack, if phase mixing is rapid enough we may assume that the beam reaches equilibrium before each transition. Transverse energy is conserved as a beam enters a bend from a straight, but since the beam acquires a finite $\Delta x x_m$ as it finds equilibrium in the bend, the transverse energy will be discontinuous entering a straight from a bend. Δx^2, $\Delta x'^2$, Δy^2, $\Delta y'^2$ are, of course, continuous at all transitions. In Fig. (2), we have applied this formulation to a small scale recirculator, which is not undergoing acceleration. The prescription above for calculation of the emittance was carried out numerically, and compared with the 3-D particle- in-cell code WARP. As can be seen, the emittance growth is tracked closely although the higher frequency behavior is not seen. (For small values of $\Delta(\delta p/p)^2$, or large values of σ/σ_0 the prescription overestimates the emittance growth, since the assumption of complete phase-mixing between transitions is not achieved.)

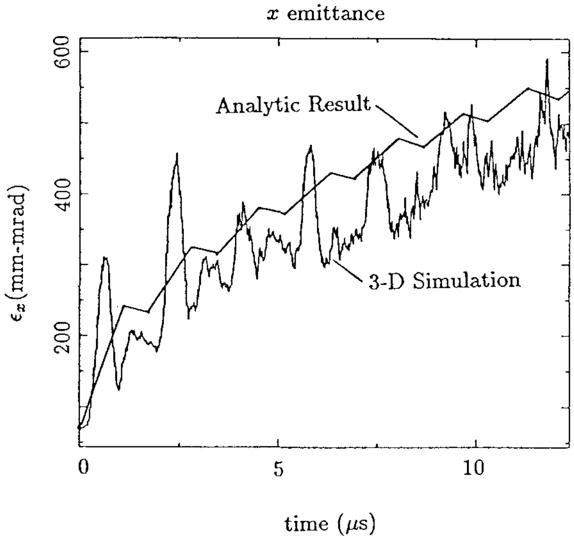

Fig. 1. Emittance growth in a racetrack. The parameters are $\sigma_0, = 72°$, $\sigma = 8°$, $\rho = 3.6$m, $Am_p\beta^2c^2/2 =$ 10MeV.

Discussion

There are several sources of emittance growth in recirculators. In ref. [2], emittance growth from misaligned quadrupoles was estimated and constraints were set on tolerances for quadrupole alignment. Emittance growth from sharp transitions, as discussed above, provides another source of emittance growth. Using the parameters of the beam at the exit of the High Energy Ring of ref. [2] leads to an emittance growth by a factor of about 2. Since the entrance beam parameters lead to a much smaller emittance growth, the normalized emittance will grow by less than a factor of 2. This is within the emittance "budget" in the design of ref. [2]. It is also possible, that the transitions between bends and straights can be made gradual enough so that equilibria are reached adiabatically, with little associated growth in the normalized emittance.

Conclusions

We have used equations of motion, in which focusing, space charge, and dispersion in a bend, are included in an approximate manner. By assuming the transverse energy of the beam is conserved, and that the beam reaches an equilibrium state, we estimated the emittance growth from beams which make transitions from bends to straight sections and vice versa. In a recirculator, in which four such transitions are made per lap, we have calculated the emittance growth under the assumption that the equilibrium state is reached between each transition. In ILSE-scale rings the analytic result agreed generally with the 3-D WARP simulation, when σ_0/σ was small and the velocity spread was sufficiently large (so that the assumption of phase mixing between transitions was realized). In the High Energy Ring of ref. [2], this prescription yielded an emittance growth by a factor of 2.

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