FRINGE FIELDS OF CURRENT-DOMINATED MULTIPOLE MAGNETS*

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Abstract

We determine analytic functions that describe the fringe field region of Lambertson, or cosine-wound, magnets. In particular, we are interested in determining the aberrations, up to fifth order, of a beam transiting our large-bore current-dominated quadrupoles. We determine the scalar potential from the vector potential calculated first for a single current loop and then for a $2N$ symmetric current loop multipole magnet.

Introduction

Figure 1 shows the geometry used for calculating the magnetic potential functions (dipole geometry is indicated in the figure). The magnet coils are placed on a constant radius (r_0) cylinder centered at the coordinate origin, with the z axis as the rotation axis and with one current loop centered about $\theta = 0$. (If the magnet has layered coils with different r_0, add the separate potential functions for each layer.) A field point is described by the coordinates (r_f, θ_f, z_f) and a source point by (r_0, θ_s, z_s). The surface current densities J, which provide the magnetic field, are assumed to result from θ-directional and z-directional current-carrying zero-radius wires.

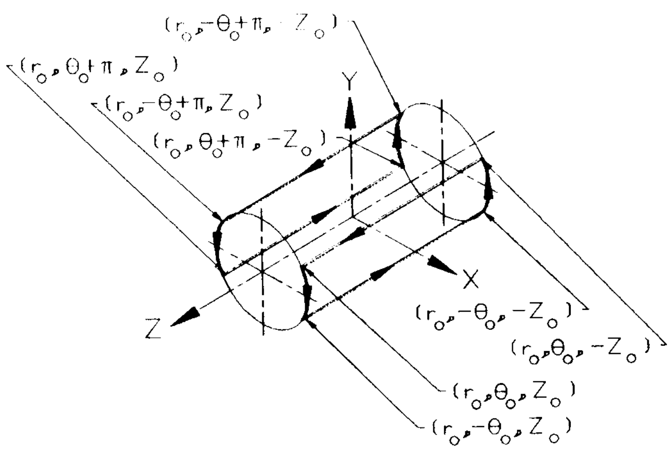

Fig. 1. Magnet coil loop orientation for dipole field.

Given that "N" is the fundamental harmonic number (1 for dipole, 2 for quadrupole, etc.), there are $2N$ current loops that have the following properties. [The coordinates of one corner of one loop are (r_0, θ_0, z_0); we ignore the constant r_0.]

$$J_z(\theta_0, z_s) = -J_z(-\theta_0, z_s) = (-1)^n J_z(\theta_0 + n\pi/N, z_s), \ (-z_0 \le z_s \le +z_0)\,; \quad (1)$$

$$J_\theta(\theta_s, z_0) = -J_\theta(\theta_s, -z_0) = (-1)^n J_\theta(\theta_s + n\pi/N, z_0), \ (-\theta_0 \le \theta_s \le +\theta_0)\,; \quad (2)$$

$$n = 1, ..., 2N-1\,.$$

These equations can be verified for the dipole geometry using Fig. 1.

We calculate the vector potential $\vec{A}_J$, that is due to the current loops, as a power series expansion in r^m and $\sin(m\theta)$ and then gauge transform $\vec{A}_J$ to $\vec{A}_V$, where the θ component of $\vec{A}_V$ is zero (this simplifies the relationship between the scalar and vector potentials). The scalar potential V is written as a power series expansion in r^m and $\sin(m\theta)$ times unknown functions, $F_m(z)$, of the z coordinate. The functions $F_m(z)$ are determined from a truncated power series for A_{V_z}. Once the scalar potential V is determined, the components of the vector potential $\vec{A}_V$ are obtained from Eqs. (3) and (22) (see below) in this gauge where $A_{V_\theta} = 0$. For each multipole component "m" in Eq. (22), we have

$$A_{m_z} = -\frac{r}{m}\frac{sin(m\theta)}{cos(m\theta)}\partial_r V_m\,,$$

$$A_{m_r} = -\frac{r}{m}\frac{sin(m\theta)}{cos(m\theta)}\partial_z V_m\,.$$

Assumptions and Formulas

A time-independent magnetic field in a source-free region can be calculated from either a vector $\vec{A}$ or scalar V potential function where

$$\vec{B} = -\nabla V = \nabla \times \vec{A} \quad (3)$$

and V satisfies Laplace's equation (the divergence of $\vec{B}$ is zero).

The magnetic field that is due to a current distribution is obtained from the vector potential

$$\vec{A}_J(\vec{r}_f) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}_s)dV_s}{|\vec{r}_f - \vec{r}_s|}\,. \quad (4)$$

[The subscript J indicates that this vector potential is calculated from the current-density distribution. The subscript V, introduced below, denotes the vector potential that is related to the scalar potential V through Eq. (3).]

Because the curl of the gradient of a scalar function is identically zero, the magnetic field is invariant under the gauge transformation

$$\vec{A}_V = \vec{A}_J + \nabla\varphi \ , \quad (5)$$

which is used to calculate φ for a gauge where $A_{V_\theta} = 0$. Because

$$A_{V_\theta} = A_{J_\theta} + \frac{1}{r}\partial_\theta\varphi = 0 \ ,$$

$$\varphi = -\int r A_{J_\theta} d\theta \ \text{ and}$$

$$\vec{A}_V = \vec{A}_J - \hat{a}_r\partial_{r_f}\int r_f A_{J_\theta} d\theta_f - \hat{a}_\theta A_{J_\theta} - \hat{k}\partial_{z_f}\int r_f A_{J_\theta} d\theta_f\,. \quad (6)$$

(The $\hat{a}$'s represent unit vectors in cylindrical coordinates).

A general form for the scalar potential that satisfies Laplace's equation (and Bessel's equation in cylindrical coordinates) and reduces to a harmonic expansion for an axially independent field is[1]

$$V(r_f, \theta_f, z_f) = \sum_{m\ge1} r_f^m \cos(m\theta_f + \theta_m) \sum_{n=0}^{\infty} \frac{m!(-r_f^2\partial_{z_f}^2)^n F_m(z_f)}{2^{2n} n!(n+m)!}\,, \quad (7)$$

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where m is an integer and θ_m is a constant phase angle that depends on the index m. [A more general form (not needed here) for the scalar potential is required if θ_m is not constant in z. Then, a separate expansion in $\sin(m\theta)$ and $\cos(m\theta)$ is necessary.] The scalar potential in Eq. (7), which consists of a power series expansion in r for each harmonic number m, is fully determined by the functions $F_m(z)$ and constant phase angles θ_m. The boundary conditions imposed on the $F_m(z)$'s are that all the derivatives of F_m vanish at $z \to \pm\infty$. Table I lists the coefficients

$$\mathcal{K}_1(m,n) = \frac{m!}{2^{2n} n!(n+m)!}$$

in Eq. (7), which indicates the rapid convergence of the power series in n.

TABLE I. FUNCTION $\mathcal{K}_1(m,n)$.

m	n = 0	1	2	3	4	5	6
1	1.0E+00	1.3E-01	5.2E-03	1.1E-04	1.4E-06	1.1E-08	6.7E-11
2	1.0E+00	8.3E-02	2.6E-03	4.3E-05	4.5E-07	3.2E-09	1.7E-11
3	1.0E+00	6.3E-02	1.6E-03	2.2E-05	1.9E-07	1.2E-09	5.6E-12
4	1.0E+00	5.0E-02	1.0E-03	1.2E-05	9.7E-08	5.4E-10	2.2E-12
5	1.0E+00	4.2E-02	7.4E-04	7.8E-06	5.4E-08	2.7E-10	1.0E-12

The minimal magnetic scalar potential symmetry condition for a perfectly constructed multipole magnet with fundamental harmonic number N is

$$V(\theta) = -V(\theta + \pi/N) \tag{8}$$

(this includes the fringe field region), which is only true when the index m in Eq. (7) is restricted to

$$m = N(2k+1)\,, \quad k = 0,1,2,\ldots\,. \tag{9}$$

The z component of the vector potential A_{V_z} must satisfy the general analytic expression obtained by combining Eqs. (3) and (7); therefore, we find that

$$A_{V_z} = -\sum_m r_f^m \sin(m\theta_f + \theta_m) \times \sum_{n=0}^{\infty} \frac{(m-1)!(m+2n)(-r_f^2 \partial_{z_f}^2/4)^n F_m(z_f)}{n!(n+m)!}\,. \tag{10}$$

We use Eq. (4) to calculate the vector potential $\vec{A}_J$ for current loops, perform a power series expansion on $\vec{A}_J$, and gauge transform $\vec{A}_J$ to $\vec{A}_V$ using Eq. (6). The $F_m(z)$'s are obtained by equating the coefficient of the lowest power of r for each $sin(m\theta + \theta_m)$ in the power series expansion for $\vec{A}_{V_z}$ to the expansion coefficients in Eq. (10). Given the $F_m(z)$'s, Eq. (7) determines V.

Several power series expansions that are needed to calculate the vector potential from the current distribution are listed below[2]:

$$\int \frac{dX}{W^m W^{1/2}} = \frac{X 2^{2m-1}(m!)^2}{W^{1/2} m(2m)!} \sum_{n=0}^{m-1} \frac{(2n)!}{(n!)^2 2^{2n} W^n}\,,$$
$$W = 1 + X^2\,; \tag{11}$$

$$(1+X)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! X^n}{2^{2n}(n!)^2}\,, \quad X < 1\,; \tag{12}$$

$$\cos^{2n+1}\theta = \frac{(2n+1)!}{2^{2n}} \sum_{k=0}^{n} \frac{\cos(2k+1)\theta}{(n+k+1)!(n-k)!}\,,$$
$$n = 0,1,2,\ldots; \tag{13}$$

$$\cos^{2n}\theta = \frac{(2n)!}{2^{2n-1}} \left[\sum_{k=1}^{n} \frac{\cos 2k\theta}{(n+k)!(n-k)!} + \frac{1}{2(n!)^2} \right],$$
$$n = 1,2,3,\ldots\,. \tag{14}$$

Calculation of the Vector Potential

In this section, $\vec{A}_J$ is calculated for current loops and is gauge transformed using Eq. (6) to determine $\vec{A}_{V_z}$.

Normalize all dimensions to the magnet coil radius r_0. Let $R_f = r_f/r_0$, where $r_f < r_0$, $Z_f = z_f/r_0$ and $Z_s = z_s/r_0$. Equation (4) becomes

$$\vec{A}_J(r_f,\theta_f,z_f) = \frac{\mu_0 r_0}{4\pi} \times \int \frac{\hat{a}_r J_\theta \sin(\theta_f - \theta_s) + \hat{a}_\theta J_\theta \cos(\theta_f - \theta_s) + \hat{k} J_z}{\left[1 + R_f^2 - 2R_f\cos(\theta_f - \theta_s) + (Z_f - Z_s)^2\right]^{1/2}} d\theta_s dZ_s\,. \tag{15}$$

The surface current densities J_θ and J_z result from a constant current I_0 in wire elements. Therefore, the current densities are the product of I_0 times the appropriate delta functions in θ and z. We rewrite Eq. (15) [using a "suggestive" notation where the $(\pm I_0)$ indicates that the appropriate current direction must be taken into account] as

$$\vec{A}_J = \frac{\mu_0}{4\pi} \sum_{wires} (\pm I_0) \times \int \frac{d\theta_s dZ_s}{\left[1 + R_f^2 - 2R_f\cos(\theta_f - \theta_s) + (Z_f - Z_s)^2\right]^{1/2}} \times [\hat{a}_r \delta(Z_s, Z_{wire}) \sin(\theta_f - \theta_s) + \hat{a}_\theta \delta(Z_s, Z_{wire}) \cos(\theta_f - \theta_s) + \hat{k}\delta(\theta_s, \theta_{wire})]\,. \tag{16}$$

A power series expansion in R_f and $\cos(\theta)$ is required for the integrand in Eq. (16). Let T represent components of the integrand in Eq. (16) and p denote an exponent ($p = 0$ corresponds to the $\hat{k}$ component, $p = 1$ corresponds to the $\hat{a}_\theta$ component, and the $\hat{a}_r$ component is not needed) and then perform the following algebraic steps:

$$T = \frac{R_f^p \cos^p(\theta_f - \theta_s)}{\left[1 + R_f^2 - 2R_f\cos(\theta_f - \theta_s) + (Z_f - Z_s)^2\right]^{1/2}}$$
$$= \frac{R_f^p \cos^p(\theta_f - \theta_s)}{\left\{\left[1 + R_f^2 + (Z_f - Z_s)^2\right]\left[1 - \frac{2R_f\cos(\theta_f - \theta_s)}{1 + R_f^2 + (Z_f - Z_s)^2}\right]\right\}^{1/2}}$$
$$= \sum_{m=0}^{\infty} \frac{(2m)! 2^m R_f^{m+p} \cos^{m+p}(\theta_f - \theta_s)}{2^{2m}(m!)^2 \left[1 + R_f^2 + (Z_f - Z_s)^2\right]^{m+1/2}}$$
$$= \sum_{m=0}^{\infty} \frac{(2m)! R_f^{m+p} \cos^{m+p}(\theta_f - \theta_s)}{2^m (m!)^2 [1 + (Z_f - Z_s)^2]^{m+1/2} \left[1 + \frac{R_f^2}{1 + (Z_f - Z_s)^2}\right]^{m+1/2}}\,, \tag{17}$$

where Eq. (12) has been used. (This expansion is valid because the expansion parameter is less than 1 in all cases.)

Equation (17) contains $\cos^{m+p}(\theta_f - \theta_s)$ terms. These terms can be converted into $\cos[(m+p)(\theta_f - \theta_s)]$ terms by using

Eqs. (13) and (14). We are interested only in the $R_f^{m+p}\cos[(m+p)(\theta_f-\theta_s)]$ terms in Eq. (17) that will be related to Eq. (10). These terms only appear when $k=n=(n+p)$ in Eqs. (13) and (14). (All other terms have the form $R_f^{m+n}\cos(m\theta)$ where $n\geq 1$.) Also,

$$\left[1+\frac{R_f^2}{1+(Z_f-Z_s)^2}\right]^{-(m+1/2)}=1-\ldots$$

and, therefore,

$$T=\sum_{m=0}^{\infty}\frac{(2m)!R_f^{m+p}\cos[(m+p)(\theta_f-\theta_s)]}{2^{2m+p-1}(m!)^2\left[1+(Z_f-Z_s)^2\right]^{m+1/2}}+\ldots. \quad (18)$$

The ellipsis in Eq. (18) refers to the nonrequired terms.

By combining Eqs. (6), (16), and (18) and considering only terms of the form $R_f^m(\cos,\sin)(m\theta)$, the following equation is obtained:

$$\begin{aligned}A'_{V_z}=&\frac{\mu_0}{2\pi}\sum_{wires}(\pm I_0)\int d\theta_s dZ_s\\&\times\sum_{m=1}^{\infty}\frac{(2m)!R_f^m}{2^{2m}(m!)^2\left[1+(Z_f-Z_s)^2\right]^{m+1/2}}\\&\times\{\delta(\theta_s,\theta_{wire})\cos[m(\theta_f-\theta_s)]\\&+\delta(Z_s,Z_{wire})(Z_f-Z_s)\sin[m(\theta_f-\theta_s)]\}\,. \quad (19)\end{aligned}$$

The full power series for A_{V_z} contains terms that are not required for calculating V. The prime on A_{V_z} indicates that the nonrequired terms are ignored. The $m=0$ term in Eq. (19) was ignored because the formalism does not handle this case.

Equation (19) is integrated over one current loop centered about $(\theta_s,z_s)=(0,0)$, with one corner at $(\theta_s,z_s)=(\theta_0,z_0)$. The current direction is given in Fig. (1). Equation (19), using Eq. (11), becomes

$$\begin{aligned}A'_{V_z}=&-\frac{\mu_0 I_0}{\pi}\sum_{m=1}^{\infty}\frac{R_f^m}{m}\sin(m\theta_f)\sin(m\theta_0)\\&\times\left\{\frac{(2m)!}{2^{2m}(m!)^2}\left[\frac{(Z_f-Z_0)}{[1+(Z_f-Z_0)^2]^{m+1/2}}-\frac{(Z_f+Z_0)}{[1+(Z_f+Z_0)^2]^{m+1/2}}\right]\right.\\&+\frac{1}{2}\sum_{n=0}^{m-1}\frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{(Z_f-Z_0)}{[1+(Z_f-Z_0)^2]^{n+1/2}}\right.\\&\left.\left.-\frac{(Z_f+Z_0)}{[1+(Z_f+Z_0)^2]^{n+1/2}}\right]\right\}\,. \quad (20)\end{aligned}$$

Equation (20) gives A'_{V_z} for a single current loop. The potential function $A'_{V_z}(2N)$ for a magnet containing $2N$ alternating in sign current loops is

$$\begin{aligned}A'_{V_z(2N-loops)}&=\sum_{n=0}^{2N-1}(-1)^n A'_{V_z}\left[\theta_f\rightarrow\left(\theta_f+\frac{n\pi}{N}\right)\right]\\&=2NA'_{V_z(single-loop)}\,, \quad (21)\end{aligned}$$

where the index m in Eqs. (20) and (21) is restricted to the values in Eq. (9).

In Table II the coefficients

$$\mathcal{K}_2(m)=\frac{(2m)!}{2^{2m}(m!)^2}\,,$$

TABLE II. FUNCTION $\mathcal{K}_2(m)$.

m	0	1	2	3	4	5	6
$\mathcal{K}_2(m)$	1.0	5.0E-01	3.8E-01	3.1E-01	2.7E-01	2.5E-01	2.3E-01

which appear in Eq. (20), are tabulated and this table indicates that all of the terms for a given m should be retained for calculating the potential function in Eq. (20). (This statement also applies to the calculation of V, below.)

Magnetic Scalar Potential

Combining Eqs. (7), (10), and (20) gives the magnetic scalar potential for a single current loop

$$\begin{aligned}V_{(\text{single-loop})}=&\frac{\mu_0 I_0}{\pi}\sum_{m\geq 1}(m-1)!\sin(m\theta_0)R_f^m\cos(m\theta_f)\\&\times\sum_{j=0}^{\infty}\frac{(-1)^j R_f^{2j}}{2^{2j}j!(j+m)!}\partial_{Z_f}^{2j}\left\{\frac{1}{2}\sum_{n=0}^{m-1}\frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{(Z_f-Z_0)}{[1+(Z_f-Z_0)^2]^{n+1/2}}\right.\right.\\&\left.-\frac{(Z_f+Z_0)}{[1+(Z_f+Z_0)^2]^{n+1/2}}\right]+\frac{(2m)!}{2^{2m}(m!)^2}\left[\frac{(Z_f-Z_0)}{[1+(Z_f-Z_0)^2]^{m+1/2}}\right.\\&\left.\left.-\frac{(Z_f+Z_0)}{[1+(Z_f+Z_0)^2]^{m+1/2}}\right]\right\}\,. \quad (22)\end{aligned}$$

(Note that $\theta_m=0$.) For a $2N$ symmetric magnet, combining Eqs. (21) and (22) gives

$$\begin{aligned}V_{(2\text{N}-\text{loops})}&=2NV_{(\text{single-loop})}\\m&=N(2k+1)\,,\ k=1,2,\ldots\,. \quad (23)\end{aligned}$$

(Remember that $R_f=r_f/r_0$, $Z_f=z_f/r_0$, $Z_0=z_0/r_0$.)

Summary

Equation (22) gives the magnetic scalar potential for a single current loop on a cylinder of radius r_0 centered about $\theta_s=0$ and $z_s=0$, with one corner at $\theta_s=\theta_0$ and $z_s=z_0$. All the lengths in Eq. (22) have been normalized to r_0 ($R_f=r_f/r_0$, $Z_f=z_f/r_0$, $Z_0=z_0/r_0$). Equation (23) gives the magnetic scalar potential for an N-multipole magnet with $2N$ symmetrically spaced current loops.

Equation (22) can be used to study the effect of errors in radial and angular current loop position and has been used to study certain errors in our Lambertson magnets.

References

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