UNDERSTANDING SCALING LAWS*

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Summary

In this paper, I show what accelerator scaling laws are, how they can be be generated, and how they are used. A scaling law is a relation between machine parameters and beam parameters. An alternative point of view is that a scaling law is an imposed relation between the equations of motion and the initial conditions. The relation between the parameters is obtained by requiring the beam to be matched. (A beam is said to be matched if the phase-space distribution function is a function of single-particle invariants of the motion.) Because of this restriction, the number of independent parameters describing the system is reduced.Using simple models for bunched- and unbunched-beam situations, I show how scaling laws determine the general behavior of beams in accelerators. Such knowledge is useful in design studies for new machines such as high-brightness linacs. The simple model I present shows much of the same behavior as the detailed RFQ model discussed by Wadlinger[1] at this conference.

Introduction

In any accelerator beam-dynamics model, there are two kinds of parameters:

(1) Machine parameters (focusing strength, frequency,...).

(2) Beam parameters (emittance, beam radius,...).

A scaling law is a relation between these two kinds of parameters. We get this relation by requiring the beam to be matched. This restriction reduces the total number of independent parameters in our model. Once we have a scaling law, we can obtain formulas that are useful in design studies. These formulas generally take the form of expressing a beam parameter in terms of some useful set of independent parameters of our machine/beam model.

In this paper I will first describe the general procedure for generating scaling laws. Then I will work out the basic scaling laws and useful formulas for two cases: a simple model for an unbunched beam and a simple model for a bunched beam.

Procedure for Generating and Using Scaling Laws

First Step: Define Dynamical System

First we establish a model of the machine and beam. There will be a certain number, n, of independent parameters $(p_1, p_2, \ldots, p_n)$ that describe the state of the machine/beam system. Different sets of n parameters may be used, some more convenient than others.

Second Step: Relate Machine to Beam Parameters

The next step is to obtain relations between the machine parameters and the beam parameters. These are the scaling laws.

We generate these relations by requiring the beam to be matched. The number, m, of such relations depends on the number of degrees of freedom and on the detail with which we describe the beam and the forces acting on it. The matching constraint reduces the number of independent parameters in the dynamical system to n-m.

$$\begin{aligned} &F_1(p_1, p_2, \ldots, p_n) = 0, \\ &\quad \cdots \\ &F_m(p_1, p_2, \ldots, p_n) = 0 \end{aligned} \qquad (1)$$

Another way to look at this is to consider the scaling law as a relation between the equations of motion and the initial conditions. The equations of motion, which describe the evolution of the 6-D phase space, depend on the machine parameters and on the spatial part of the beam distribution. These equations can have many solutions, depending on the initial beam, which is described by the 6-D phase-space distribution. The scaling law is an imposed relation between the parameters of the equations of motion and the initial conditions, which restricts the solutions to correspond to matched beams only.

Third Step: Derive Useful Formulas

Starting with the basic scaling laws given by Eq. (1), we can solve for any parameter p_i in terms of any other of the n-m parameters. (In practice, we may not be able to do this analytically, but we can always generate the implicitly defined function numerically.) An example of such a formula is

$$p_1 = p_1(p_2, p_3, \ldots, p_{n-m+1}) \,. \qquad (2)$$

This formula would be useful if, for example, we wanted to know how p_1 behaved when p_2 was held fixed and p_3 was increased. In general, if we are to obtain meaningful conclusions, we must conform to the following rules. First, we must make sure that the number of parameters appearing on the right-hand side of the formula is equal to n-m. Then, we must make sure that all parameters that we wish to observe, to fix, or to vary appear explicitly in the formula.

For example, consider a four-parameter model in which the machine and beam are described by the following parameters: focusing strength, emittance, beam radius, and beam current. Suppose just one relation is required to specify a matched beam. Then, our formulas would express one of the parameters in terms of the other three. We could use these formulas to answer questions like the following: What happens to the current when the emittance is held fixed and the focusing strength is varied? (The answer will depend on the value of the beam radius.)

If some of the parameters we wish to investigate are not in the original set that we used in the scaling law, then we must transform the parameters. Suppose we wish to obtain some parameter q as a function of the parameters $(q_1, q_2, \ldots, q_{n-m})$ instead of the p_i. In this case, we use the definitions of the new parameters to eliminate, in Eq. (1), n-m+1 of the original parameters in favor of the new ones. Then we can solve for q in terms of the q_i. This process can be repeated for any number of new parameters q. In practice, only a few of the parameters differ from the p_i; therefore, the amount of algebra is not large (but because the transformations may be nonlinear, numerical solution may be required).

Example for Unbunched Cylindrical Beam

For the first example, consider a beam transporting a current I traveling in the z-direction. We assume the focusing forces are transverse, linear, independent of time and z, and that the force constant is the same in both transverse directions. We assume that the beam is an infinitely long circular cylinder of radius R with uniform charge density and that all the particles are traveling at the same velocity v.

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Equations of Motion for Unbunched Cylindrical Beam

The transverse single-particle motion is given by

$$\frac{dx}{dt} = \frac{p_x}{m}, \text{ and} \tag{3}$$

$$\frac{dp_x}{dt} = -(k_{ex} + k_{sc})x \ , \tag{4}$$

where x is a transverse displacement, p_x is the conjugate momentum, k_{ex} is the external-force constant, and k_{sc} is the space-charge-force constant. A similar equation holds for the y-direction. The space-charge force constant is (electromagnetic quantities are in mks units)

$$k_{sc} = -\frac{eI}{2\pi\varepsilon_o R^2 \upsilon} \ . \tag{5}$$

Invariants for the equations of motion are

$$A_x(x,y,p_x,p_y) = x^2 + \left(\frac{p_x}{m\omega}\right)^2 = const., \text{ and} \tag{6}$$

$$A_y(x,y,p_x,p_y) = y^2 + \left(\frac{p_y}{m\omega}\right)^2 = const. \ , \tag{7}$$

where

$$\omega = \left[\frac{k_{ex} + k_{sc}}{m}\right]^{1/2} \ . \tag{8}$$

A_x and A_y are the oscillation amplitudes in the two transverse directions.

Basic Scaling Law for Unbunched Beam Case

The beam is described by the distribution function $f(x,y,p_x,p_y,t)$, which is the density in 4-D phase space. The beam is matched if the distribution function is a function of the single-particle invariants. In our case, this means the density in phase space is constant on the invariant ellipses given by Eqs. (6) and (7).

We will describe the matched beam by four parameters: normalized emittance ε, current I, radius R, and velocity υ. The normalized emittance in the x-direction is defined to be the area occupied by the beam in x-p_x phase-space projection, divided by πmc. The boundary of a matched phase-space distribution is the same as that of an invariant ellipse. (Because our example is time independent, the invariant ellipse is also a phase-space trajectory.) Therefore, for a matched beam, we have

$$\varepsilon = \frac{x_{max}\, p_{max}}{mc} \ ; \tag{9}$$

then by using these Eqs. (6) and (7), we get the basic scaling law

$$c\varepsilon = R^2\omega \ , \tag{10}$$

where we identified the beam radius R with the maximum beam amplitude. In terms of our basic set of five parameters $(\varepsilon, I, k_{ex}, R, \upsilon)$, the scaling law is

$$\varepsilon = \frac{R^2}{c}\left[\frac{k_{ex}}{m} - \frac{eI}{2\pi\varepsilon_o R^2 \upsilon m}\right]^{1/2} \ . \tag{11}$$

Notice that our system is described by five parameters: one describing the machine (k_{ex}), and four describing the beam $(\varepsilon, I, R, \upsilon)$. The scaling law, Eq. (11), is one condition; thus, only four parameters can be independently chosen.

Formulas for Unbunched Cylindrical Beam

First, let us take the scaling law, Eq. (11), and solve for the current.

$$I = \frac{2\pi\varepsilon_o}{e}\left(k_{ex} - \frac{\varepsilon^2 mc^2}{R^4}\right)R^2\upsilon \ . \tag{12}$$

This says that, given any emittance, we can have as large a current as desired if we are willing to let the beam radius, focusing strength, or beam velocity get large enough. Of course, physical constraints will lead to practical current limits.

Now let us transform the parameters to get other useful formulas. The space-charge parameter is defined as

$$\mu = -\frac{k_{sc}}{k_{ex}} \ . \tag{13}$$

It is related to the tune depression factor as follows:

$$\frac{\sigma}{\sigma_o} = (1-\mu)^{1/2} \ . \tag{14}$$

where σ and σ_0 are the phase advances per unit independent variable with and without space charge, respectively. We can eliminate R in the basic scaling law, Eq.(11), in favor of μ to get a relation involving the $(\varepsilon, I, k_{ex}, \mu, \upsilon)$ parameters and solve for I. Similarly, we can eliminate I in Eq. (11) in favor of μ to get a relation involving the $(\varepsilon, R, k_{ex}, \mu, \upsilon)$ parameters and solve for R. This will give us formulas of the following form.

$$I = I(\varepsilon, k_{ex}, \mu, \upsilon), \text{ and} \tag{15a}$$

$$R = R(\varepsilon, k_{ex}, \mu, \upsilon) . \tag{15b}$$

The actual formulas are as follows.

$$I = \frac{2\pi\varepsilon_o\left(mc^2\right)^{1/2}}{e}\,\frac{\varepsilon\mu k_{ex}^{1/2}\upsilon}{(1-\mu)^{1/2}}, \text{ and} \tag{16}$$

$$R = \left(mc^2\right)^{1/4}\varepsilon^{1/2}\left[\frac{1}{k_{ex}(1-\mu)}\right]^{1/4} . \tag{17}$$

Suppose we fix the machine parameter k_{ex} and the beam emittance and velocity. The beam current and radius are plotted as functions of the space-charge parameter in Fig. 1. For a given machine, many different beams of a given emittance and velocity will be matched. The graph shows that beams having large μ values transport more current. Actual machines will have a finite aperture, thus, the larger beam size at larger μ values will determine a finite current limit. Also, we may want to keep the tune depression from becoming too large (perhaps we fear instabilities). Then the restriction on μ may determine the actual current limit.

Example for Bunched Beam

Let us consider the simplest possible model for a structure that transports a bunched beam. Let us assume the beam to consist of uniformly charged spherical bunches moving at a velocity υ and spaced so that they go by a fixed point in space at frequency f. Assume the focusing forces to be linear, isotropic, and constant in the beam frame of reference.

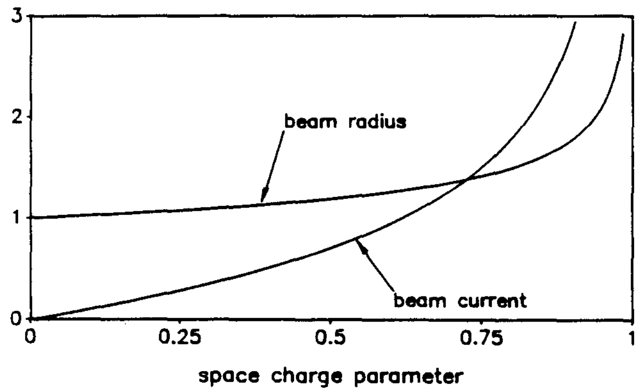

Fig. 1. Beam current and radius as a function of space-charge parameter for the unbunched, cylindrical-beam case.

Equations of Motion for Bunched Beam

The equations of motion are again given by Eqs. (1) and (2) as in the unbunched case, but the space-charge force constant is now given by the following:

$$k_{ex} = -\frac{eI}{4\pi\varepsilon_o f R^3} , \qquad (18)$$

where f is the frequency at which the bunches pass.

Basic Scaling Law for Bunched Beam Case

The basic scaling law is again determined from Eq. (9). Using the new expression for the space-charge-force constant, we get the following basic scaling law in the $(\varepsilon, f, I, k_{ex}, R)$ parameters:

$$\varepsilon = \frac{R^2}{c}\left[\frac{k_{ex}}{m} - \frac{eI}{4\pi\varepsilon_o f R^3 m}\right]^{1/2} . \qquad (19)$$

Formulas for Bunched-Beam Case

Using the same procedure of eliminating one or more of the five parameters in favor of new ones and solving for one parameter in terms of the other four, we can write formulas of the following form:

$$I = I(\varepsilon, f, k_{ex}, \mu) , \qquad (20a)$$

$$R = R(\varepsilon, f, k_{ex}, \mu) , \text{ and} \qquad (20b)$$

$$\sigma = \sigma(\varepsilon, f, k_{ex}, \mu) . \qquad (20c)$$

The parameter σ is the phase advance per period, taking into account space-charge forces. The actual formulas are as follows.

$$I = \frac{4\pi\varepsilon_o}{e}(mc^2)^{3/4} \varepsilon^{3/2} f k_{ex}^{1/4} \frac{\mu}{(1-\mu)^{3/4}} , \qquad (21)$$

$$R = (mc^2)^{1/4} \frac{\varepsilon^{1/2}}{\left[k_{ex}(1-\mu)\right]^{1/4}} , \text{ and} \qquad (22)$$

$$\sigma = \frac{1}{f}\left[\frac{k_{ex}(1-\mu)}{m}\right]^{1/2} . \qquad (23)$$

Figure 2 shows the current, radius, and phase advance plotted as a function of the frequency for fixed values of ε, k_{ex}, and μ. We see that the current increases with frequency but that the beam radius is independent of frequency. Because the available accelerator aperture decreases with frequency, there is a frequency at which the pipe is filled, which corresponds to the current limit for the given values of ε, k_{ex}, and μ. The fact that higher frequencies are favored for high-brightness machines has been recognized for some time, both from scaling law[2] and particle simulation[3] work. In this very simple model, the R parameter is also the bunch half-length. Therefore, the maximum allowable R-value also depends on velocity, with larger velocities being favored. The limit occurs at the point where adjacent bunches overlap. In real machines such as DTL and RFQ linacs, the bunch-length limitation is caused by the nonlinearity of the longitudinal focusing forces; thus, the real limit occurs before the overlapping bunch limit.

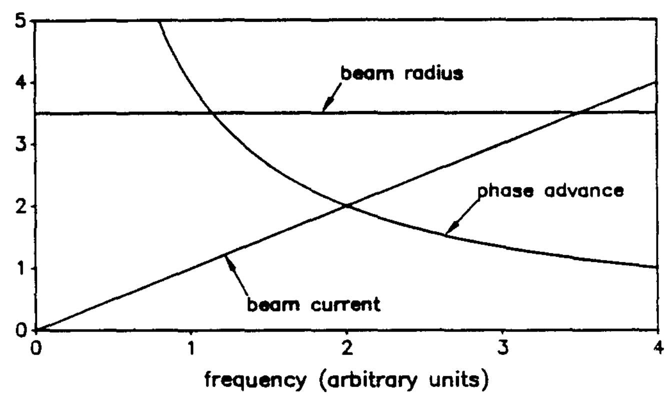

Fig. 2. Beam current, beam radius, and phase advance per period as a function of the frequency for the bunched-beam case. The ε, k_{ex}, and μ parameters are held fixed.

We also see that high-brightness machines tend to have a small phase advance per period. Of course, high-brightness machines need strong focusing forces, but strong focusing means a large phase advance per unit time. The phase advance per period is small because the frequency is high. Because of the small σ, high-brightness alternating-gradient structures will have a small flutter factor. This means the space-charge forces will be practically time independent. This fact may be helpful in developing simulation codes for high-brightness structures.

It is encouraging to see that this very simple bunched-beam model has the same qualitative behavior as seen by Alan Wadlinger in his scaling law study using a realistic model of the RFQ. His paper[1] shows how RFQs can be designed using scaling laws.

References

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