

THE INFLUENCE OF EQUIPARTITIONING ON THE EMITTANCE OF INTENSE CHARGED-PARTICLE BEAMS*

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Summary

We combine the ideas of kinetic energy equipartitioning and nonlinear field energy to obtain a quantitative description for rms emittance changes induced in intense beams with two degrees of freedom. We derive equations for emittance change in each plane for continuous elliptical beams and axially symmetric bunched beams, with arbitrary initial charge distributions within a constant focusing channel. The complex details of the mechanisms leading to kinetic energy transfer are not necessary to obtain the formulas. The resulting emittance growth equations contain two separate terms: the first describes emittance changes associated with the transfer of energy between the two planes; the second describes emittance growth associated with the transfer of nonlinear field energy into kinetic energy as the charge distribution changes.

Introduction

Recently, we presented a differential equation^{1,2} for continuous round beams with continuous linear focusing (a smoothed representation of a real transport line), which expresses a relationship between the rate of change of rms emittance and the rate of change of the nonlinear field energy. The nonlinear field energy is the residual field energy possessed by beams with nonuniform charge distributions. It depends only on the shape of the charge distribution and corresponds to the field energy available for emittance growth. Using approximations valid for a space-charge-dominated beam, namely constant rms beam size and homogenization (uniformity) of the final charge density, the integrated differential equation yields an expression for emittance growth that agrees well with the numerical simulations.^{1,2} The emittance growth formula also agrees with a formula proposed earlier³ to explain numerical simulation results for a quadrupole transport channel. Equivalent forms of the emittance and field-energy differential relation had been discovered earlier,^{4,5} but it appears that the utility of this result for obtaining a better understanding of emittance growth effects in linacs and transport systems had not been recognized. Experimental evidence for the importance of the emittance-growth equation for unneutralized beams in a real quadrupole transport channel has also been reported.^{6,7}

For the round symmetric beam, a single emittance-growth mechanism was isolated, characterized by a rapid charge-density redistribution, as the charged beam particles, behaving like a plasma, adjust their positions to shield the external field from the interior of the beam. For linear focusing, this implies a uniform charge density to produce the required linear space-charge field for exact shielding of beams in the extreme space-charge (zero-emittance) limit. In general, this is an approximation because finite emittance beams tend toward a matched charge density with a central uniform core and a finite thickness

boundary, roughly equal to the Debye length. The rms emittance growth itself arises from the nonlinear space-charge fields, when the beam charge density is nonuniform. The basic relationship between field energy and rms emittance is not restricted to a round continuous beam. The relationship for a 1-D sheet beam was derived earlier^{8,9} and, more recently, one of us (I. H.) has generalized the differential equation to include asymmetric continuous (elliptical) beams and bunched beams in free space.¹⁰ This new result allows us to derive more general formulas for space-charge-induced emittance growth that include (1) the charge density redistribution and (2) kinetic energy exchange between different degrees of freedom.

The suggestion of emittance growth associated with kinetic energy exchange was made many years ago to explain the numerical studies for the CERN and Brookhaven linac injectors.¹¹⁻¹⁴ The law of equipartition of energy was invoked, which asserts that in thermal equilibrium, the same average kinetic energy is associated with each degree of freedom. But further study was required to establish this principle in a charged-particle accelerator, where collective fields dominate over particle collisions. More recent numerical simulation studies of high-current asymmetric continuous beams^{15,16} and bunched linear-accelerator beams¹⁷⁻¹⁹ reaffirmed the importance of the kinetic-energy exchange mechanism. Detailed analysis of the K-V distribution for 2-D asymmetric beams^{15,16} resulted in a prediction of coherent-mode instability thresholds for asymmetric beams, which established a collective field mechanism for kinetic-energy exchange. The predicted threshold values even agreed closely with numerical simulation results for bunched beams.^{19,20} Simulation studies clearly showed that equipartitioning does occur when the space-charge forces become large.²⁰ However, theoretical predictions for the magnitude of the emittance growth from equipartitioning were still not obtainable.

In this paper, we use the general equation relating field energy and rms emittance to derive equations for emittance growth for rms-matched beams with continuous linear focusing, which include both the charge density redistribution and the kinetic-energy exchange mechanisms. These equations contain two final-state parameters: the final nonlinear field energy and the final value of a new quantity called the partition parameter. We discuss the characteristics of these final-state parameters deduced from our numerical simulation studies. We invoke two hypotheses, observed from simulation studies to approximately characterize the final state for space-charge-dominated beams: (1) homogenization (charge-density uniformity) and (2) equipartitioning. For intense beams, these hypotheses allow us to obtain values for the two final-state parameters and equations for emittance growth that depend only on the initial beam properties. We present equations for both 2-D continuous beams and axially symmetric bunched beams. We present details of the numerical simulation results for the 2-D asymmetric beams in an accompanying paper²¹ at this conference. The equations will apply to beams in free space, or beams within a conducting pipe whose radius is much larger than the beam size. The effects of image charges in smaller pipes will require further study.

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2-D Continuous Beams

Emittance Growth

For a general 2-D continuous beam, the relationship between field energy and rms emittance, exact for an elliptical beam with an arbitrary charge distribution in free space, can be written as¹⁰

$$\frac{1}{X^2} \frac{d\epsilon_x^2}{ds} + \frac{1}{Y^2} \frac{d\epsilon_y^2}{ds} = -K \frac{dU_n}{ds}, \quad (1)$$

where s is the distance along the beam axis, and the parameters X and Y are the total semiaxes of the equivalent uniform beam (uniform beam with the same rms sizes as the actual beam), related to the rms beam sizes $\sqrt{x^2}$ and $\sqrt{y^2}$ by $X = 2\sqrt{x^2}$ and $Y = 2\sqrt{y^2}$. For nonelliptical beams, Eq. (1) is an approximation that can be tested by numerical studies. The quantity K is the generalized perveance given in terms of charge e , mass m , number of beam particles per unit length N_b , velocity v , relativistic mass factor γ , and free-space permittivity ϵ_0 by $K = e^2 N_b / 2\pi \epsilon_0 m v^2 \gamma^3$, and the actual beam current I is given by $I = N_b e v$. The quantity $U_n = U/w_0$ is the normalized nonlinear field energy, where U is the difference between the self-electric field energies per unit length of the actual beam and the equivalent uniform beam, and the quantity $w_0 = (eN_b)^2 / 16\pi\epsilon_0$ is a field energy per unit length normalization parameter. Both the electric- and magnetic-field contributions are contained in Eq. (1) by including the factor γ^3 in the definition of K (γ^2 accounts for the magnetic field and γ accounts for the relativistic mass). We have learned from our numerical studies²¹ that U_n is independent of X and Y , independent of beam current, and has a unique value for a given charge-density profile that is a measure of the charge-density nonuniformity. The minimum value of U_n is $U_n = 0$ for a uniform beam. Values of U_n for some common charge distributions in Table I and in Refs. 1 and 2 illustrate these properties. The rms emittance, ϵ_x , in Eq. (1) is defined by

$$\epsilon_x = 4(\overline{x^2} \overline{x'^2} - \overline{xx'})^{1/2}, \quad (2)$$

where the beam divergence x' is related to the x -velocity component \dot{x} by $\dot{x} = vx'$, and similar definitions apply for the y -plane.

TABLE I
 U_n FOR SOME COMMON 2-D DISTRIBUTIONS

Distribution	Charge density	U_n
Gaussian	$\exp(-x^2/2x_0^2 - y^2/2y_0^2)$	0.154
Parabolic (waterbag)	$1 - x^2/x_0^2 - y^2/y_0^2$	0.0224
Uniform (K-V and thermal)	1	0.000
Hollow	$x^2/x_0^2 + y^2/y_0^2$	0.0754

We assume that a beam, with arbitrary initial charge density profile and arbitrary kinetic energies in the x - and y -planes, is injected into a linear continuous focusing channel and transforms from an initial to a final rms matched state. We allow for unequal kinetic energies and unequal focusing in the x - and y -planes. We will assume that the rms beam sizes are assumed to remain constant as the beam propagates, a good approximation for rms-matched,

space-charge-dominated beams with linear continuous focusing, as we have learned from numerical simulation work.²¹ Then we can integrate Eq. (1) and obtain

$$\frac{\Delta\epsilon_x^2}{X^2} + \frac{\Delta\epsilon_y^2}{Y^2} = -K\Delta U_n. \quad (3)$$

Equation 3 is the basis for the emittance-growth equations that we will derive.

For an rms matched beam, we can write $\epsilon_x = XX'$ and $\epsilon_y = YY'$, where X' and Y' are related to rms beam divergences $\sqrt{x'^2}$ and $\sqrt{y'^2}$ by $X' = 2\sqrt{x'^2}$ and $Y' = 2\sqrt{y'^2}$. It is convenient to introduce a new parameter P that we call the partition parameter, defined by

$$P = X'^2/Y'^2. \quad (4)$$

When the transverse motion is nonrelativistic, the quantity P is a measure of the kinetic-energy asymmetry. Using the partition parameter, we can re-express Eq. (3) in the two forms that give emittance-growth equations in each plane. We obtain

$$\frac{\epsilon_{xf}}{\epsilon_{xi}} = \left[1 - \frac{(P_i - P_f)}{P_i(1 + P_f)} - \frac{P_f}{(1 + P_f)} \frac{KX^2}{\epsilon_{xi}^2} (U_{nf} - U_{ni}) \right]^{1/2}, \quad (5)$$

and

$$\frac{\epsilon_{yf}}{\epsilon_{yi}} = \left[1 + \frac{(P_i - P_f)}{(1 + P_f)} - \frac{1}{(1 + P_f)} \frac{KY^2}{\epsilon_{yi}^2} (U_{nf} - U_{ni}) \right]^{1/2}, \quad (6)$$

where the subscripts i and f refer to the initial and final states. It is convenient to rewrite Eqs. (5) and (6) using the results²²

$$\frac{KY^2}{\epsilon_{yi}^2} = G_2(X/Y) \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right), \quad (7)$$

and

$$\frac{KX^2}{\epsilon_{xi}^2} = \frac{G_2(X/Y)}{P_i} \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right), \quad (8)$$

where the function $G_2(X/Y)$ is defined as

$$G_2(X/Y) = \frac{1}{2} (1 + X/Y). \quad (9)$$

The quantities k_{yi} and k_{0y} are the initial betatron wave number (tune) of the equivalent uniform beam (including space charge), and the zero-current betatron wave number, respectively, for the y -plane. The betatron tune ratio in the x -plane can be easily expressed in terms of the y -plane ratio²² and could equally well have been used instead. These betatron wave numbers measure the effectiveness of the focusing with and without space charge and as beam intensity increases, k_{yi}/k_{0y} decreases. The relationship between the tune-depression ratio k_{yi}/k_{0y} and the beam and channel parameters K , ϵ_x , ϵ_y , k_{0x} , and k_{0y} , is algebraically complicated and is most easily obtained numerically, but near the space-charge limit, a simple result is²²

$$\frac{k_y}{k_{0y}} = \left(\frac{\epsilon_y k_{0y}}{2K} \right) \left(\frac{k_{0x}^2 + k_{0y}^2}{k_{0x}^2} \right). \quad (10)$$

We can also write $P_i = (\epsilon_x/\epsilon_y)^2/(X/Y)^2$, where $X/Y \approx k_{0y}^2/k_{0x}^2$, for a space-charge-dominated beam. Substitution of Eqs. (7) and (8) into Eqs. (5) and (6) yields

$$\frac{\epsilon_{xf}}{\epsilon_{xi}} = \left[1 - \frac{(P_i - P_f)}{P_i(1 + P_f)} - \frac{P_f}{P_i(1 + P_f)} G_2(X/Y) \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) (U_{nf} - U_{ni}) \right]^{1/2}, \quad (11)$$

and

$$\frac{\epsilon_{yf}}{\epsilon_{yi}} = \left[1 + \frac{(P_i - P_f)}{(1 + P_f)} - \frac{1}{(1 + P_f)} G_2(X/Y) \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) (U_{nf} - U_{ni}) \right]^{1/2}. \quad (12)$$

Equations (11) and (12) express the emittance-growth ratios in terms of the initial beam variables X/Y , P_i , k_{yi}/k_{0y} , and U_{ni} and two final-state variables P_f and U_{nf} . The equations contain two growth (or decay) terms: one that depends on the change in the partition parameter P and the other that depends on the change in nonlinear field energy U_n . For the case of a round symmetric beam, where $X = Y$ and $P_f = P_i = 1$, these equations reduce to the results already presented in Refs. 1 and 2.

The values of the final-state parameters must be determined either from additional theory or from numerical simulation. At present, we have the numerical simulation results available to us.²¹ From these studies using different initial distributions, we conclude that the charge-density redistribution is very rapid, and after only a few plasma periods, the beam density, ignoring a low-density halo, is nearly uniform ($U_{nf} \approx 0$) when the beam intensity is high enough that the charge-density redistribution effect causes significant emittance growth.

Kinetic energy exchange is a slower process than the charge density redistribution and can typically, take tens of plasma periods. For the following discussion, we will assume that $P_i > 1$. The final partition parameter, P_f , depends strongly on k_{yi}/k_{0y} . For reasonable propagation distances, we can identify three distinct regions: (1) a stable region, where P does not change (tune depressions above approximately $k_{yi}/k_{0y} \approx 0.5$ to 0.6 for most initial distributions); (2) a transition region with partial or incomplete equipartitioning, where $1 < P_f < P_i$ (k_{yi}/k_{0y} below the tune depression threshold); and (3) full equipartitioning, where $P_f = 1$ for sufficiently low tune depressions. As the beam propagates further, $P_f \rightarrow 1$ throughout the transition region, and the curve of P_f versus k_{yi}/k_{0y} approaches a step function.

If we assume that for the highest beam intensities the final beam charge density is uniform or homogenized ($U_{nf} = 0$) and that the final kinetic energy is equipartitioned ($P_f = 1$), we obtain

$$\frac{\epsilon_{xf}}{\epsilon_{xi}} = \left[1 - \frac{(P_i - 1)}{2P_i} + \frac{G_2(X/Y)}{2P_i} \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) U_{ni} \right]^{1/2}, \quad (13)$$

and

$$\frac{\epsilon_{yf}}{\epsilon_{yi}} = \left[1 + \frac{(P_i - 1)}{2} + \frac{G_2(X/Y)}{2} \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) U_{ni} \right]^{1/2}. \quad (14)$$

Recent experimental studies⁶ show evidence that kinetic-energy exchange effects do lead to emittance growth in real unneutralized 2-D asymmetric beams with quadrupole focusing.

Minimum Final Emittance

A general result for a minimum final emittance can be derived. This is most easily done by returning to Eqs. (5) and (6) and using the results,²² near the extreme space-charge limit, that $X^2 = 2K(k_{0y}/k_{0x})^2/(k_{0x}^2 + k_{0y}^2)$ and $Y^2 = 2K(k_{0x}/k_{0y})^2/(k_{0x}^2 + k_{0y}^2)$.

Then, using the same assumptions of final-state homogenization and equipartitioning, we obtain

$$\epsilon_{xf}^2 = \epsilon_{xi}^2 \frac{(1 + P_i)}{2P_i} + \frac{(k_{0y}/k_{0x})^2}{k_{0x}^2 + k_{0y}^2} K^2 U_{ni}^2, \quad (15)$$

and

$$\epsilon_{yf}^2 = \epsilon_{yi}^2 \frac{(1 + P_i)}{2} + \frac{(k_{0x}/k_{0y})^2}{k_{0x}^2 + k_{0y}^2} K^2 U_{ni}^2. \quad (16)$$

The minimum final emittances correspond to initial emittances $\epsilon_{xi} = \epsilon_{yi} = 0$ (the extreme space-charge limit). Then we obtain

$$\epsilon_{xf,min} = \sqrt{\frac{k_{0y}/k_{0x}}{k_{0x}^2 + k_{0y}^2}} K U_{ni}^{1/2}, \quad (17)$$

and

$$\epsilon_{yf,min} = \sqrt{\frac{k_{0x}/k_{0y}}{k_{0x}^2 + k_{0y}^2}} K U_{ni}^{1/2}. \quad (18)$$

Equations (17) and (18) predict that the minimum final emittance depends on the initial nonlinear field energy U_{ni} , but not on the initial partition parameter P_i . The minimum final emittances are linearly proportional to beam current through the parameter K . The numerical simulation results for a round symmetric beam^{1,2} have shown excellent agreement with Eqs. (15) and (16), and supporting experimental evidence has also been reported⁶ for unneutralized beams in quadrupole focusing channels.

Scaling With Beam and Channel Parameters

Using the pair of matched rms envelope equations for the x - and y -planes, and the definitions of the equivalent, uniform beam tunes k_x and k_y , it is straightforward to show that the three variables (P , X/Y , and k_y/k_{0y} that determine the emittance growth) can be expressed as functions of three new dimensionless variables that depend directly on the beam and channel parameters K , ϵ_x , ϵ_y , k_{0x} and k_{0y} .²² These three new variables consist of two current-dependent parameters, $u_x = K/2\epsilon_x k_{0x}$ and $u_y = K/2\epsilon_y k_{0y}$, and the zero-current beam-aspect ratio $X_0/Y_0 = (\epsilon_x k_{0y}/\epsilon_y k_{0x})^{1/2}$. Thus, for a given initial charge-density profile, constant values of u_x , u_y , and X_0/Y_0 should produce the

same emittance growth from charge-density redistribution and kinetic-energy exchange. We note that the ratios I/ϵ_x and I/ϵ_y enter into u_x and u_y to determine the emittance growth.

Axially Symmetric Bunched Beams

Emittance Growth

For an axially symmetric bunch in free space with rms semiaxes $a = \sqrt{x^2}$, $b = \sqrt{y^2}$ and $b = \sqrt{z^2}$ in the laboratory frame, we write the differential relation between the field energy and rms-emittance for an arbitrary charge-density profile as¹⁰

$$\frac{2}{a^2} \frac{d\epsilon_y^2}{ds} + \frac{1}{b^2} \frac{d\epsilon_z^2}{ds} = - \frac{32}{mv^2 \gamma^3 N ds} (W - W_U) \quad (19)$$

where W is the space-charge electric-field energy of the bunch, and W_U is the same quantity for the equivalent, uniform ellipsoidal bunch. Equation (19) is exact only for a uniform ellipsoid, and is an approximation for other cases. In Eq. (19), complete symmetry is assumed for the transverse x - and y -planes. The quantity N is the number of beam particles in the bunch, and the emittance definitions are as given by Eq. (2) with the effective z -plane divergence given by $z' = (z - v)/v$, where z is the laboratory velocity component of each particle along the beam axis, and v is the velocity of the center of mass of the bunch. We assume that a beam bunch is injected into a channel with linear continuous focusing in all three planes with equal focusing in x and y (a smoothed representation of a well-bunched beam in a linac). We allow for unequal kinetic energies and unequal focusing in the transverse and longitudinal planes, where the longitudinal kinetic energy is defined in the rest frame of the bunch. The beam bunch is assumed to transform from an initial to a final state, and as in the 2-D case, the rms beam sizes are assumed to remain constant as the beam propagates, a good approximation for space-charge-dominated beams with linear continuous focusing. Integrating Eq. (19), we obtain

$$\frac{2\Delta\epsilon_y^2}{a^2} + \frac{\Delta\epsilon_z^2}{b^2} = \frac{16K_3}{b} G_0(b/a) \Delta U_n \quad (20)$$

where K_3 is a perveance-like parameter with dimensions of length given by

$$K_3 = \frac{Ne^2}{20\sqrt{5}\pi\epsilon_0 mv^2 \gamma^3} \quad (21)$$

The average beam current for a string of bunches in an rf linac with one bunch per rf period is given by $I = Nec/\lambda$, where c is the speed of light, and λ is the rf wavelength. The function $G_0(b/a)$ is given by

$$G_0(b/a) = (1 - M) + M(b/a)^2 \quad (22)$$

where M is the ellipsoid form factor,²³ which is approximately $M = 1/(3 b/a)$ for a nearly spherical bunch and is $M = 1/3$ for an exactly spherical bunch. The quantity $U_n = (W - W_U)/w_3$ is the normalized nonlinear field energy, where we have chosen to normalize to w_3 given by

$$w_3 = \frac{(Ne)^2 G_0(b/a)}{40\sqrt{5}\pi\epsilon_0 b} \quad (23)$$

For a spherical bunch w_3 equals the space-charge electric-field energy within an equivalent uniform bunch.

In Table II, we show values of U_n calculated for a spherical bunch for some common distributions. For a spherical bunch, we find that U_n depends only on the shape of the charge distribution and is independent of rms beam size and beam current; also, U_n is a measure of charge-density nonuniformity, having a minimum value of zero for a uniform bunch. Additional work will be required to determine whether this choice of normalization results in a dimensionless nonlinear field energy that is a function only of the charge-density profile in the general ellipsoidal case, independent of the beam semiaxes a and b .

TABLE II

U_n FOR COMMON SPHERICAL BUNCH DISTRIBUTIONS

Distribution	Charge density	U_n
Gaussian	$\exp(-r^2/2\sigma^2)$	0.308
Parabolic	$1 - r^2/R^2$	0.0368
Uniform	1	0.00

Again, we introduce the partition parameter P , defined by

$$P = b'^2/a'^2, \text{ where } b' = \sqrt{z'^2} \text{ and } a' = \sqrt{x'^2} \quad (24)$$

In the nonrelativistic limit, which is the case of most interest P is a measure of the kinetic energy asymmetry between the longitudinal and transverse planes in the bunch rest frame. Using the partition parameter, we can re-express Eq. (20) to give emittance growth equations in each plane. The results are

$$\frac{\epsilon_{zf}}{\epsilon_{zi}} = \left[1 - \frac{2}{P_i} \frac{(P_i - P_f)}{(2 + P_f)} \right. \quad (25)$$

$$\left. - \frac{1}{P_i} \frac{P_f}{(2 + P_f)} G(b/a) \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) (U_{nf} - U_{ni}) \right]^{1/2}$$

and

$$\frac{\epsilon_{yf}}{\epsilon_{yi}} = \left[1 + \frac{(P_i - P_f)}{(2 + P_f)} \right. \quad (26)$$

$$\left. - \frac{1}{(2 + P_f)} G(b/a) \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) (U_{nf} - U_{ni}) \right]^{1/2}$$

where the subscripts i and f refer to the initial and final states. The function $G(b/a)$ is defined by

$$G(b/a) = 2G_0(b/a)/3(1 - M) \quad (27)$$

and the quantities k_{yi} and k_{0y} are the initial betatron wave number of the equivalent uniform beam with space charge and the zero-current betatron wave number, respectively, for the y -plane. The initial tune-depression ratio k_{zi}/k_{z0} can be expressed²⁴ as a function of k_{yi}/k_{0y} , P_i , and b/a . An approximate relationship for the tune-depression ratio k_{yi}/k_{0y} , with

respect to the parameters of the beam and channel valid near the extreme space-charge limit, is²⁴

$$\frac{k_y}{k_{0y}} = \frac{\epsilon_y}{4k_{0y}} \left(\frac{k_{0z}^2}{K_3} \right)^{2/3} \left(\frac{1 + 2k_{0y}^2/k_{0z}^2}{3} \right)^{4/3} \quad (28)$$

We can also write $P_i = (\epsilon_{zi}/\epsilon_{yi})^2 / (b/a)^2$, where $b/a \approx (1 + 2k_{0y}^2/k_{0z}^2)/3$ for a space-charge-dominated beam.

As in the 2-D case, if we assume that for high beam intensities the beam approaches final-state homogenization ($U_{nf} = 0$) and kinetic-energy equipartitioning ($P_f = 1$ nonrelativistically), we obtain emittance-growth equations given by

$$\frac{\epsilon_{zf}}{\epsilon_{zi}} = \left[1 - \frac{2}{3} \frac{(P_i - 1)}{P_i} + \frac{G(b/a)}{3P_i} \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) U_{ni} \right]^{1/2} \quad (29)$$

and

$$\frac{\epsilon_{yf}}{\epsilon_{yi}} = \left[1 + \frac{(P_i - 1)}{3} + \frac{G(b/a)}{3} \left(\frac{k_{0y}^2}{k_{yi}^2} - 1 \right) U_{ni} \right]^{1/2} \quad (30)$$

Numerical simulation studies for the case of spherical bunches with different charge density distributions⁹ have shown good agreement with Eqs. (29) and (30). The above emittance-growth equations for nonspherical bunches have not yet been tested by numerical simulation.

Minimum Final Emittance

The assumptions of final-state homogenization and equipartitioning near the extreme space-charge limit result in

$$\epsilon_{zf}^2 = \epsilon_{zi}^2 \frac{(2 + P_i)}{3P_i} + \frac{16G_z(b/a)}{3} \left(\frac{K_3^2}{k_{oz}^2} \right)^{2/3} U_{ni} \quad (31)$$

and

$$\epsilon_{yf}^2 = \epsilon_{yi}^2 \frac{(2 + P_i)}{3} + \frac{16G_y(b/a)}{3} \left(\frac{K_3^2}{k_{0y}^2} \right)^{2/3} U_{ni} \quad (32)$$

where $G_y(b/a) = G_0[3(1 - M)/2(b/a)^4]^{1/3}$ and $G_z(b/a) = G_0[3(b/a)^2M]^{1/3}$. The minimum final emittance occurs when the initial emittances are $\epsilon_{yi} = \epsilon_{zi} = 0$ (extreme space-charge limit), which results in

$$\epsilon_{zf, \min} = 4 \left(\frac{G_z}{3} \right)^{1/2} \left(\frac{K_3^2}{k_{oz}^2} \right)^{1/3} U_{ni}^{1/2} \quad (33)$$

and

$$\epsilon_{yf, \min} = 4 \left(\frac{G_y}{3} \right)^{1/2} \left(\frac{K_3^2}{k_{0y}^2} \right)^{1/3} U_{ni}^{1/2} \quad (34)$$

Equations (33) and (34) depend on the initial nonlinear field energy U_{ni} , but are independent of P_i , a conclusion that was found also for the 2-D problem. For the bunched-beam problem, the minimum final emittances are proportional to $I^{2/3}$, through the parameter K_3 . This is in contrast to the linear dependence obtained for the 2-D continuous beam. The $I^{2/3}$ prediction agrees well with numerical simulation studies

for a spherical bunch.⁹ The only linac experimental results of which we are aware¹¹ are unpublished, and the reported result is that the measured scaling of minimum final emittance is in the range of $I^{1/3}$ to $I^{1/2}$. If these results are confirmed, they suggest that other emittance-growth mechanisms may also contribute significantly in real linacs.

Scaling With Beam and Channel Parameters

From the matched envelope equations for the bunched beam and the definitions of the equivalent uniform bunch tunes k_y and k_z , we can show²⁴ that the three variables (P , b/a , and k_y/k_{0y}) that determine the emittance growth can be expressed as functions of three new variables that depend directly on the beam and channel parameters K_3 , ϵ_y , ϵ_z , k_{0y} , and k_{0z} . The three new variables are u_y , u_z , and b_0/a_0 , where $u_y = 4K_3 k_{0z}^{1/2} / \epsilon_y \epsilon_z k_{0y}$, $u_z = 4K_3 k_{0y}^{1/2} / \epsilon_z \epsilon_y k_{0z}$, and $b_0/a_0 = (\epsilon_z k_{0y} / \epsilon_y k_{0z})^{1/2}$. Then, constant values of u_y , u_z , and b_0/a_0 should yield the same emittance growth ratios for a given initial charge-density profile, assuming that no other sources of emittance growth are present. For bunched beams, the ratios $I/\epsilon_y \epsilon_z^{1/2}$ and $I/\epsilon_z \epsilon_y^{1/2}$ enter into u_y and u_z and determine the emittance growth, in contrast to the I/ϵ ratios that enter for 2-D beams. For spherical bunches, this implies that the emittance growth depends on $I/\epsilon^{3/2}$. We have found excellent agreement of this scaling rule in our numerical simulation studies for spherical bunches.⁹

Conclusions

We have presented equations for 2-D continuous beams and axially symmetric bunched beams with linear continuous focusing in free space that predict space-charge-induced emittance growth associated with two mechanisms: (1) charge-density redistribution and (2) kinetic-energy exchange. We have used the relation between field energy and rms emittance to obtain results, which have been expressed in terms of both the initial beam variables, and two dimensionless final-state variables: (1) the final, normalized, nonlinear field energy U_{nf} , a measure of the final charge-density nonuniformity and (2) the final partition parameter P_f , a measure of the final kinetic energy asymmetry.

In the absence of additional theory to predict these final variables from a given initial state, we have used numerical simulation studies to characterize the final beam. Thus far, we have done numerical studies for 2-D continuous beams^{1,2,21} and spherical bunched beams.⁹ We find that the assumption of a final, uniform charge density ($U_{nf} = 0$), which, neglecting beam-halo contributions, is expected in the extreme space-charge limit, leads to a good first approximation for the rapid charge-density redistribution component of emittance growth. For 2-D beams, a general characterization of the final partition parameter, which determines the usually slower, kinetic-energy exchange process, is more complicated. When $P_i > 1$, three regions of initial tune depression are identified: (1) a stable region at high initial tune depression, $k_{yi}/k_{0y} > 0.5$ to 0.6 , where P is unchanged, (2) a transition region, where P_f has approached equipartitioning, $1 < P_f < P_i$, and (3) a fully equipartitioned region, where $P_f = 1$. As the beam propagates further, P_f approaches unity throughout the transition region. Therefore, final equipartitioning of the beam is observed within a range of initial tune depressions that begins at $k_{yi}/k_{0y} = 0$, and approaches a stability threshold near $k_{yi}/k_{0y} = 0.5$ to 0.6 with increased distance of beam propagation.

In addition to the emittance-growth equations, we have derived formulas for minimum final emittances

corresponding to an extreme space-charge limit, when $\epsilon_j = 0$. We predict that the minimum final emittances should vary with beam current as $\epsilon_{f,\min} \propto I$ for 2-D continuous beams, and as $\epsilon_{f,\min} \propto I^{2/3}$ for bunched beams.

Finally, we have shown how space-charge-induced emittance growth scales with the beam and channel parameters through two dimensionless current-dependent parameters and the aspect ratio of the zero-current beam. We predict that emittance growth depends on I/ϵ_x and I/ϵ_y for 2-D continuous beams and on $I/\epsilon_x \epsilon_z^{1/2}$ and $I/\epsilon_y \epsilon_z^{1/2}$ for bunched beams. Experimental evidence for emittance growth from charge redistribution of unneutralized beams in real quadrupole transport channels has been reported.^{6,7} Further studies will be needed for real quadrupole systems before we can evaluate the utility of these predictions for improved high-current beam transport and accelerator design.

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