NUMERICAL STUDIES OF EMITTANCE EXCHANGE IN 2-D CHARGED PARTICLE BEAMS*

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Summary

We describe results obtained from a two- dimensional particle-following computer code that simulates a continuous, nonrelativistic, elliptical charged-particle beam with linear continuous focusing. Emittances and focusing strengths can be different in the two transverse directions. The results can be applied, for example, for a quadrupole transport system in a smooth approximation to a real beam with unequal emittances in the two planes. The code was used to study emittance changes caused by kinetic-energy exchange between transverse directions and by shifts in charge distributions. Simulation results for space-charge-dominated beams agree well with analytic formulas reported in these proceedings.[1] From simulation results, an empirical formula was developed for a "partition parameter" (the ratio of kinetic energies in the two directions) as a function of initial conditions and beamline length. Quantitative emittance changes for each transverse direction can be predicted by using this parameter. Simulation results also agree with Hofmann's generalized differential equation[2] relating emittance and field energy.

Simulation Techniques

The code follows trajectories of individual macroparticles (line charges) as they are influenced by linear continuous focusing and space-charge forces in a circular pipe with perfectly conducting walls. The line charges are actually cylindrical charge clouds with radius chosen large enough to minimize false collisional effects but small enough to reproduce (as closely as possible) structure in the beam's charge distribution.[3] Cartesian coordinates are used with z being the longitudinal (beam axis) direction, and x and y the transverse directions. For problems in this study, free-space beams are simulated by using a large pipe radius; average beam radius is 1 cm, pipe radius is 50 cm. Space-charge forces are calculated by summing individual particle-to-particle forces and are applied as impulses to individual particles at short z-intervals. Electric field energy is calculated in two ways: first, by calculating potentials at the position of each charge and summing qV over the charges; second, by calculating x and y electric-field components and evaluating

$$\frac{\varepsilon_o}{2}\int E_x^2\,dv \quad \text{and} \quad \frac{\varepsilon_o}{2}\int E_y^2\,dv$$

over a grid filling the pipe volume, thereby obtaining the x- and y-components of the total field energy. Results of the two methods agree well. The second method is useful in studying energy conservation and exchange between x- and y-directions. Arbitrary initial particle distributions can be set up with different tunes and focusing strengths in x and y. Initially, beams are matched to the channel in an rms sense; that is, the second moments are stationary. For most of the problems run for this study, 1000 macroparticles were used.

Comparison of Simulation and Theory

The essence of the theoretical work described in Ref. 1 is that for beams with elliptical symmetry and linear continuous focusing there are two important space-charge mechanisms affecting emittance. The first is adjustment of the beam's charge distribution to match external focusing forces. This redistribution of charge, occurring within about one-fourth plasma period, results in transfer of nonlinear field energy[4] (field energy in excess of that generated by an equivalent uniform beam with the same second moments as the real beam) to kinetic energy of beam particles. The second mechanism, usually operating more slowly, is transfer of kinetic energy from one coordinate direction to another, resulting in partial or complete kinetic energy equipartitioning. While superficially similar to equipartitioning of energy in a gas, this transfer does not result from individual particle collisions but, presumably, from interactions between the individual particles and collective fields. Hofmann has shown[5,6] that coherent-mode instabilities can lead to kinetic-energy exchange. For Kapchinskii-Vladimirskii (K-V) beams, these instabilities occur only below sharply defined tune thresholds that depend on tune depression and ratios of emittances and tunes; for tunes above this threshold, there are no significant instabilities and the beam is stable.

In Ref. 1, both of these mechanisms have been included in emittance-growth equations that represent the ratios of final to initial emittances for given conditions. For a 2-D continuous beam, these equations are

$$\frac{\varepsilon_{xf}}{\varepsilon_{xi}} = \left[1 - \frac{(P_i - P_f)}{P_i(1 + P_f)} - \frac{P_f}{2P_i(1 + P_f)}\left(1 + \frac{X}{Y}\right)\left(\frac{k_{0y}^2}{k_{yi}^2} - 1\right)\left(U_{nf} - U_{ni}\right)\right]^{1/2} \tag{1}$$

and

$$\frac{\varepsilon_{yf}}{\varepsilon_{yi}} = \left[1 + \frac{(P_i - P_f)}{(1 + P_f)} - \frac{1}{2(1 + P_f)}\left(1 + \frac{X}{Y}\right)\left(\frac{k_{0y}^2}{k_{yi}^2} - 1\right)\left(U_{nf} - U_{ni}\right)\right]^{1/2} . \tag{2}$$

The subscripts i and f represent initial and final states. The parameter $P = X'^2/Y'^2$ is the ratio of mean kinetic energies in the x- and y-directions, which we call the partition parameter. For this study, we have adopted the convention that $P_i \geq 1$; in other words, x is the plane of higher initial kinetic energy.

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$X = 2\sqrt{\overline{x^2}}$, $Y = 2\sqrt{\overline{y^2}}$, $X' = 2\sqrt{\overline{x'^2}}$, and $Y' = 2\sqrt{\overline{y'^2}}$

are the position and divergence envelopes of the equivalent K-V beam. The equivalent beam envelopes X and Y are assumed constant, a good assumption for space-charge-dominated beams. The zero-current betatron wave number in the y-direction is k_{0y}, and k_y is the wave number with space charge of the equivalent K-V beam (we could also have expressed the formula in terms of k_{0x} and k_x); U_n is the normalized nonlinear electric-field energy per unit length.[1,4] We found numerically that U_n is dependent only on the charge distribution for an elliptical beam and not on beam current, area πXY, or aspect ratio X/Y.

Figure 1 shows emittances, X'^2 and Y'^2 (proportional to transverse kinetic energy), and nonlinear electric-field energy for simulation of an initially Gaussian beam with initial values $\varepsilon_x/\varepsilon_y = 4$, $k_x/k_{0x} = 0.3$, $k_y/k_{0y} = 0.131$, k_x/k $k_x$eq 2.0, $P_i = 8.0$. In the first one-fourth plasma period, nonlinear field energy decreases; transverse kinetic energies and emittances increase. We interpret this effect as charge-redistribution emittance growth. Subsequently, there is transfer of kinetic energy from x to y with equipartitioning occurring at about three plasma periods. Over 550 simulations were run, covering initial conditions for K-V, Gaussian, nonstationary waterbag,[4] and thermal (semi-Gaussian) beams with $(\varepsilon_x/\varepsilon_y)_i = 4$, including ranges from $k_x/k_y = 0.4$ to 4.1, $k_x/k_{0x} = 0.1$ to 1. Also, K-V beams with $(\varepsilon_x/\varepsilon_y)_i = 1$ were run with the same initial tune ratios.

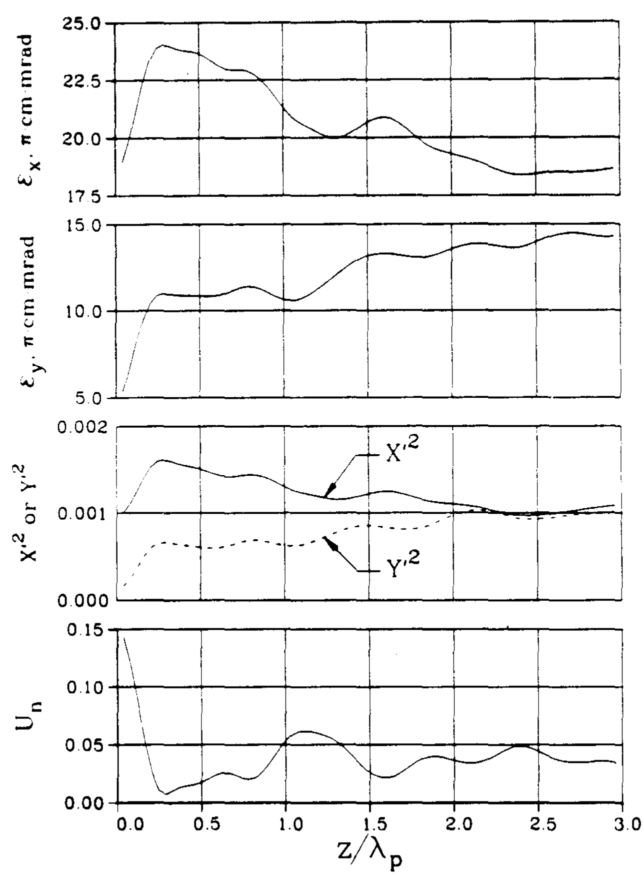

Fig. 1. Variation of beam parameters vs number of plasma periods for initially Gaussian beam, $P_i = 8$, $k_{xi}/k_{0x} = 0.3$, $(\varepsilon_x/\varepsilon_y)_i = 4$, $X/Y = 1.41$.

In Fig. 2, we compare the predictions of Eqs. (1) and (2) with the simulation results for all the problems that were run to $z/\lambda_p = 16$, where z is distance along the beamline and λ_p is the plasma wavelength of the equivalent uniform beam,

$$\lambda_p = 2\pi v \sqrt{\frac{\varepsilon_o m}{ne^2}} \quad .$$

The data points are the ratios of emittance growth from the simulations to growth predicted by Eqs. (1) and (2), using the values of P_f and U_{nf} obtained from the simulations. Each data point is a separate computer run, with a unique combination of P_i, initial emittance ratio, tune and charge distribution. The points are plotted versus k_{yi}/k_{0y} for convenience. As a test of Eqs. (1) and (2), this plot shows the agreement between the analytic expressions for emittance growth and the simulation results, when P_f and U_{nf} are known.

In Fig. 2 and in later figures, the y-emittance points show more scatter than the x-points because for most of the problems, the initial y-emittance is less than the initial x-emittance; therefore, the influence of "noise" (statistical scatter, etc.) is proportionately greater for the y-points.

In general, without additional theory or numerical simulation, only the initial state of the beam will be known. Equations (1) and (2) will not be useful unless one can predict U_{nf} and P_f. After one-fourth plasma period, taking $U_{nf} = 0$ is usually a good assumption for getting emittance growth,[4] because (1) charge redistribution occurs rapidly; (2) if space charge dominates, the beam rapidly becomes approximately uniform with a tail of about the Debye length at the edge of the beam; therefore, U_{nf} will be small; (3) if space charge is weak, the factor k_{0y}^2/k_{yi}^2-1 will be small, and the kinetic-energy exchange term will dominate. On the other hand, P_f is more difficult to predict. Simulations show the following behavior: For a given P_i, initial distribution, and number of plasma periods, if we plot P_f versus initial tune k_y/k_{0y}, three distinct regions are revealed. First, above a certain threshold in k_y/k_{0y}, the beam is stable, and $P_f = P_i$ for a uniform beam; for nonuniform beams above this

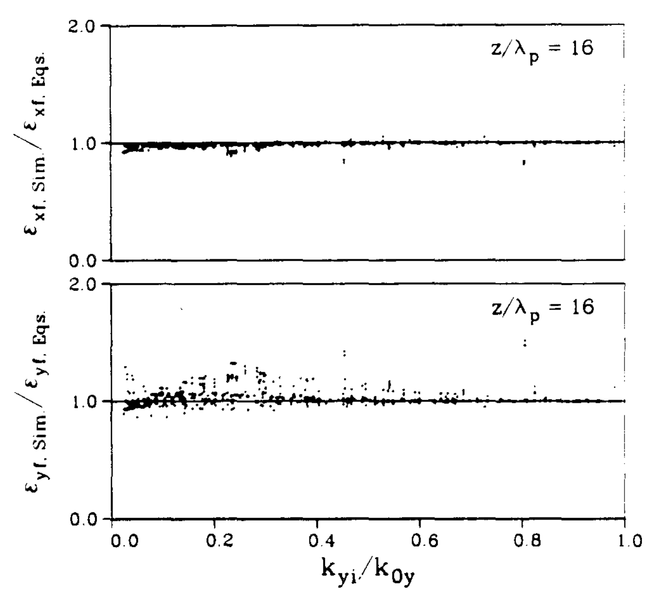

Fig. 2. Comparing simulation to Eqs.(1) and (2); U_{nf} and P_f from simulations.

threshold, there is charge redistribution but no x-y energy transfer. This charge redistribution lowers P even though there is no kinetic energy exchange between x and y. Second, for k_y/k_{0y} far below the threshold, after a few plasma periods $P_f = 1$; kinetic energy has completely equipartitioned for all initial charge distributions. Third, between these regions is a transition region characterized by $1 < P_f < P_i$. If the beams are followed for a larger number of plasma periods, the threshold remains about the same, but the equipartition region grows at the expense of the transition region; the slope of the curve of P_f versus k_y/k_{0y} increases in this region. This behavior is shown in Figs. 3 and 4, with simulation data plotted for various initial charge distributions for 4 and 32 plasma periods.

For some purposes, it may be useful to have a rough estimate of P_f. By fitting the simulations for K-V beams with $(\varepsilon_x/\varepsilon_y)_i = 4$, we arrive at an approximate formula for P_f after z/λ_p plasma periods:

$$P_f = 1 + \frac{P_i - 1}{1 + t} \quad , \tag{3}$$

where t is a dimensionless parameter given by

$$t = 0.01\, e^{\left[-12\, (z/\lambda_p)^{0.22} \left(\frac{k_i}{k_0} - 0.55\right)\right]} \quad . \tag{4}$$

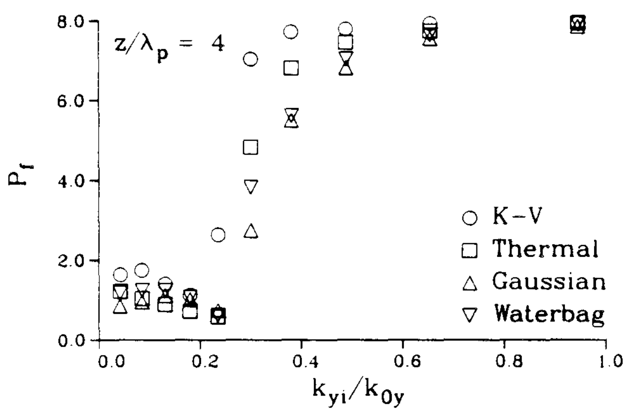

Fig. 3. Comparing different initial distributions at $z/\lambda_p = 4$; $(\varepsilon_x/\varepsilon_y)_i = 4$, $X/Y = 1.41$.

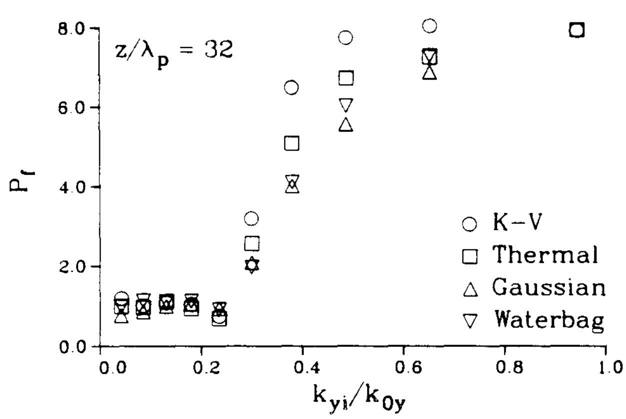

Fig. 4. Comparing different initial distributions at $z/\lambda_p = 32$; $(\varepsilon_x/\varepsilon_y)_i = 4$, $X/Y = 1.41$.

The initial tune depression k_i/k_0 in Eq. (4) is the lesser of $k_{x,i}/k_{0,x}$ or $k_{y,i}/k_{0,y}$. We choose the form of this equation to describe some features of the P_f dependence observed in the simulations. These features are (1) $P_f = P_i$ for k_i/k_0 greater than a threshold of approximately 0.55 and (2) for $k_i/k_0 < 0.55$, P_f approaches 1 (equipartitioning) at a rate that depends on how far the tune depression is below the threshold. For highly space-charge-dominated beams (k_i/k_0 far below threshold), equipartitioning occurs in only a few plasma periods. The formula is represented by dashed lines in Fig. 5. It provides a fairly accurate estimate of P_f for most of the problems for which it was fitted, that is, K-V beams with $(\varepsilon_x/\varepsilon_y)_i = 4$. For other initial conditions and charge distributions, it is less accurate but still represents the data well enough to be useful for a rough approximation.

In predicting emittance growth using Eqs. (1) and (2) where P_f and U_{nf} are unknown, we assume that (1) U_{nf} is zero as discussed earlier; (2) either $P_f = 1$ (total equipartitioning) or alternatively, P_f is estimated by Eq. (3). Figures 6, 7, and 8 show comparisons between final emittances predicted by Eqs. (1) and (2) and those resulting from the simulations. These figures include all problems that were run to the specified z/λ_p. In Figs. 6 and 7 (z/λ_p to, respectively, 8 and 32 plasma periods), we assume equipartitioning with $P_f = 1$.

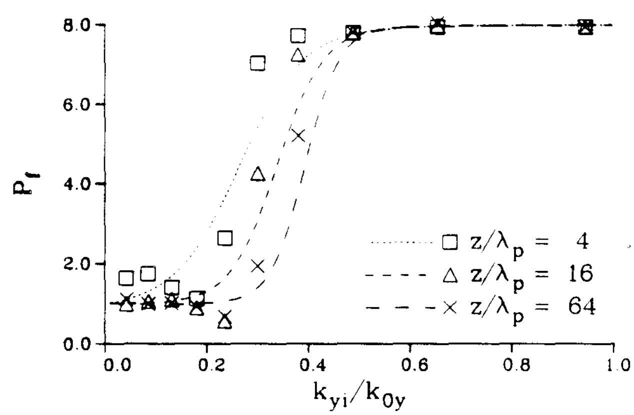

Fig. 5. K-V beams; simulations (data points) and curves from Eq. (3) (dotted lines); $(\varepsilon_x/\varepsilon_y)_i = 4$, $X/Y = 1.41$.

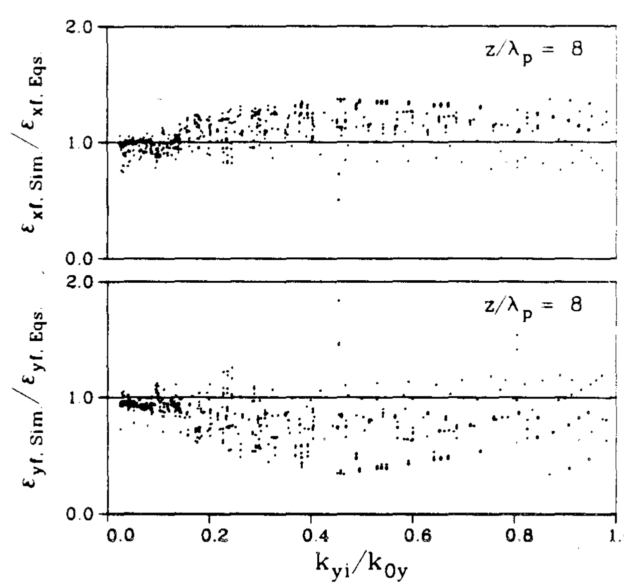

Fig. 6. Comparing simulations to Eqs. (1) and (2); $U_{nf} = 0$, $P_f = 1$.

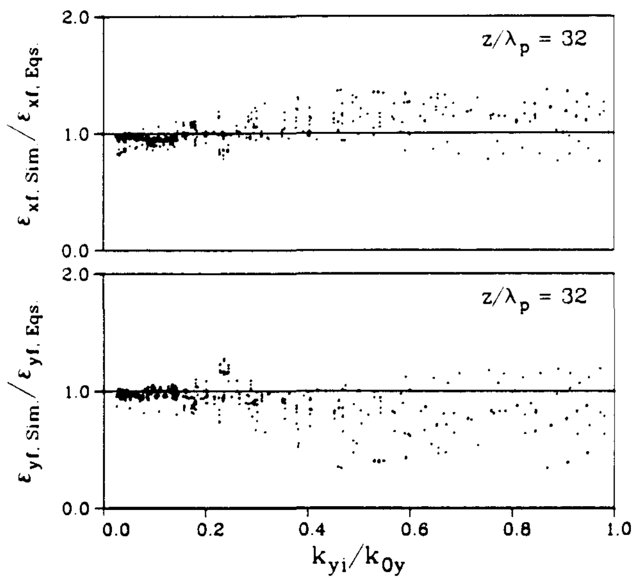

Fig. 7. Same comparison as Fig. 6, but at $z/\lambda_p = 32$.

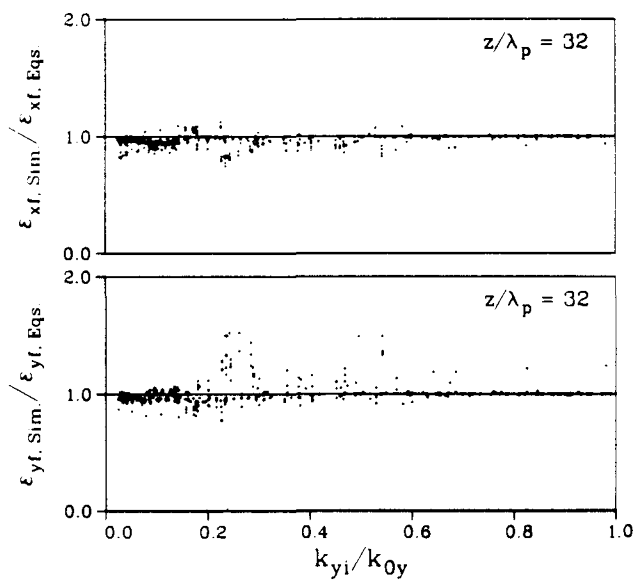

Fig. 8. Comparing simulations to Eqs. (1) and (2); $U_{nf} = 0$, P_f from Eq. (3).

As can be seen, this assumption is not accurate for emittance-dominated beams but is very good in highly space-charge-dominated beams, particularly after many plasma periods. If the beam is not space-charge dominated, then one must estimate P_f more carefully. In Fig. 8, P_f is estimated using Eq. (3). Here the agreement between the simulations and Eqs. (1) and (2) is fairly good even for emittance-dominated beams, although Eq. (3) was not fitted to all the simulations and has not been adjusted for charge redistribution in the initially nonuniform beams.

In Eq. (3), we do not attempt to reproduce details of threshold behavior, which we observe in the simulations and which are shown in Hofmann's mode charts of Refs. 5 and 6. Our simulations are consistent with the coherent-mode thresholds calculated by Hofmann and, in addition, show evidence of modes not present in the charts. For initial tune depressions close to Hofmann's thresholds, the simulations sometimes show rather complicated behavior. In some cases, the beam seems to be attracted to integer or half-integer ratios of k_x/k_y, which may result in $1 < P_f < P_i$ (partial kinetic energy exchange), in $P_f < 1$ (overpartitioning), or even in $P_f > P_i$ (kinetic-energy transfer from the lower-energy plane to the higher). These effects deserve further study; they are most apparent in K-V beams but are also observed with other initial charge distributions.

Conclusions

The agreement between the computer simulations and Eqs. (1) and (2) over a very wide range of parameters strongly supports the validity of the analytic work. In addition, Eqs. (1) and (2) have yielded formulas for predicting emittance growth in two-dimensional continuous ion beams with elliptical symmetry and linear continuous focusing, where the beams are either clearly emittance dominated or space-charge dominated. For intermediate cases, we can estimate the partition parameter P_f from the fit to the simulation data, Eq. (3), and then use it in Eqs. (1) and (2), but more theoretical work on equipartitioning rates is needed. Bunched-beam formulas[1] similar to Eqs. (1) and (2) must be tested by simulation with realistic quadrupole focusing, and may be valuable for linac design, especially if rates of equipartitioning can be more precisely determined.

Acknowledgments

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References

1. T. P. Wangler, F. W. Guy, and I. Hofmann, "The Influence of Equipartitioning on the Emittance of Intense Charged Particle Beams," these proceedings.

2. I. Hofmann, "Emittance Growth," these proceedings.

3. H. Okuda and C. K. Birdsall, "Collisions in a Plasma of Finite-Size Particles," Phys. Fluids, **13**, 2123 (1970).

4. T. P. Wangler, K. R. Crandell, R. S. Mills, and M. Reiser, "Relation Between Field Energy and RMS Emittance in Intense Particle Beams," IEEE Trans. Nucl. Sci., **32** (5), 2196 (1085); T. P. Wangler, K. R. Crandall, R. S. Mills, and M. Reiser, "Field Energy and Emittance in Intense Particle Beams," Proc. Workshop on High Current, High Brightness, High Duty Factor Ion Injectors, San Diego, May 21-23, 1985, AIP Conference Proceedings No. 139, 133 (1986).

5. I. Hofmann, "Emittance Growth of Beams Close to the Space Charge Limit," IEEE Trans. Nucl. Sci., **28** (3), 2399 (1981).

6. I. Hofmann and I. Bozsik, "Computer Simulation of Longitudinal-Transverse Space Charge Effects in Bunched Beams," in Proc. 1981 Linear Accelerator Conf., Los Alamos National Laboratory report LA-9234-C, 116 (1981).