RADIATION OF A CHARGE IN A PERFECTLY CONDUCTING CYLINDRICAL PIPE WITH A JUMP IN ITS CROSS SECTION*

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1. Introduction

The problem of finding the electromagnetic fields created by a charge moving through various geometric structures bounded by metallic walls has an important bearing on the accelerator theory. Apart from evident need for reliable evaluation of the energy loss into higher modes, there is also an important problem of evaluating the coupling impedance due to changes in the particle environment. While for a long bunch (in comparison to the relevant dimensions of the considered structure) there are several reliable numerical codes which do the job, they rapidly become too time consuming for short bunches.

Here we present the results of calculations of electromagnetic fields radiated by a point charge moving on an axis of a cylindrical superconducting pipe with an abrupt change in its cross section. From them then we find the longitudinal coupling impedance. The geometry and the coordinate system (cylindrical) is sketched in Fig. 1. For certainty we consider the case of a charge coming out of the bigger pipe of the cross section radius a and entering the narrow pipe of the cross section radius b. The opposite case of a charge exiting from the narrow pipe and entering the bigger one can be considered in a similar way. Coupling impedance of a cross section change for a planar geometry was considered in Ref. 1.

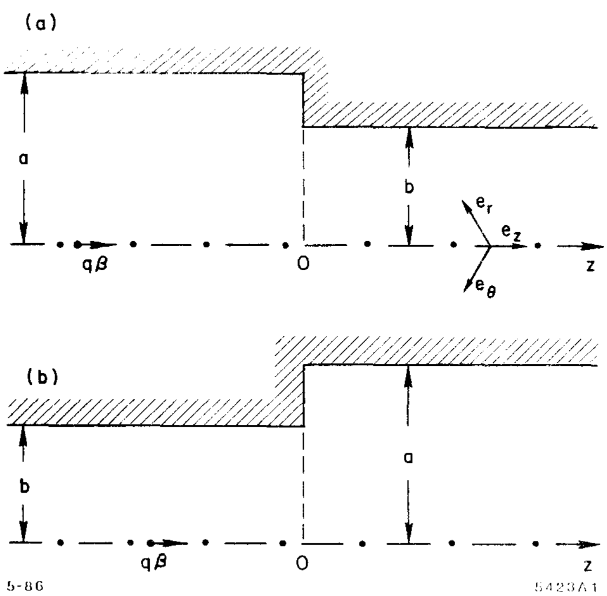

Fig. 1. Geometry of the problem and the coordinate system: (a) Incoming charge, (b) Outgoing charge.

We see three reasons for conducting the present work. First, it is useful to consider a problem theoretically since it gives better understanding of details of the radiation process for given geometry. Second, the numerical results obtained here are in a sense complimentary to purely numerical results of existing codes, providing an answer in the parameter regions which can not be reached by existing codes. Third, the results may be interesting in themselves sometimes. For example, high frequency instabilities within bunches depend on the average coupling impedance over a broad frequency band. Hence a simple summation over an independent cross section jumps is a rather good approximation in such cases.

Besides, the case of a pipe with a flange, which is an interesting case in itself, can be obtained from our results in the limit $a \to \infty$. The approach developed here can also serve as a starting point for investigating other geometries (such as a cylindrical scraper, for example).

The Fourier components of the current density of a point charge moving on the axis of the pipe is

$$\tilde{\mathbf{j}} = \mathbf{e}_r 0 + \mathbf{e}_\theta 0 + \mathbf{e}_z \frac{q\delta(r)}{2\pi r} \exp\left(\frac{ikz}{\beta}\right) , \tag{1}$$

where $c\beta$ is the charge velocity, $\delta(r)$ is Dirac's radial δ-function.

Due to symmetry of the problem electromagnetic field has only three nonzero components: E_r, E_z, and H_θ. The Fourier components of the solutions of the Maxwell equations which satisfy the boundary condition $E_z(z) = 0$ on the pipe wall for $z > 0, r = b$ and the radiation condition at $z \to \infty$ (conversely, condition $E_z(z) = 0$ on the pipe wall for $z < 0, r = a$ and the radiation condition at $z \to -\infty$) are known.[2] It is convenient to introduce the following notations:

$$M = qk/\pi c\gamma^2\beta^2$$

$$G_1(r,d) = K_1(\tau r) + I_1(\tau r)K_0(\tau d)/I_0(\tau d) \tag{2}$$

$$G_0(r,d) = K_0(\tau r) - I_0(\tau r)K_0(\tau d)/I_0(\tau d) \tag{3}$$

where $d = a$ or b, then

$$\tilde{E}_r^+ = \gamma M G_1(r,b)\exp\left(\frac{ikz}{\beta}\right) - i\Sigma_n B_n^+ \left(\frac{\nu_n}{b}\right) J_1\left(\frac{\nu_n r}{b}\right)\lambda_{bn}\exp(iz\lambda_{bn}) \tag{4}$$

$$\tilde{E}_z^+ = -iMG_0(r,b)\exp\left(\frac{ikz}{\beta}\right) + \Sigma_n B_n^+ \left(\frac{\nu_n^2}{b^2}\right) J_0\left(\frac{\nu_n r}{b}\right)\exp(iz\lambda_{bn}) \tag{5}$$

$$\tilde{H}_\theta^+ = \gamma\beta M G_1(r,b)\exp\left(\frac{ikz}{\beta}\right) - ik\Sigma_n B_n^+ \left(\frac{\nu_n}{b}\right) J_1\left(\frac{\nu_n r}{b}\right)\exp(iz\lambda_{bn}) \tag{6}$$

$$\tilde{E}_r^- = \gamma M G_1(r,a)\exp\left(\frac{ikz}{\beta}\right) + i\Sigma_n B_n^- \left(\frac{\nu_n}{a}\right) J_1\left(\frac{\nu_n r}{a}\right)\lambda_{an}\exp(-iz\lambda_{an}) \tag{7}$$

$$\tilde{E}_z^- = -iMG_0(r,a)\exp\left(\frac{ikz}{\beta}\right) + \Sigma_n B_n^- \left(\frac{\nu_n^2}{a^2}\right) J_0\left(\frac{\nu_n r}{a}\right)\exp(-iz\lambda_{an}) \tag{8}$$

$$\tilde{H}_\theta^- = \gamma\beta M G_1(r,a)\exp\left(\frac{ikz}{\beta}\right) - ik\Sigma_n B_n^- \left(\frac{\nu_n}{a}\right) J_1\left(\frac{\nu_n r}{a}\right)\exp(-iz\lambda_{an}) \tag{9}$$

Here $k = \omega/c$, $\tau = k/\gamma\beta$, $\gamma = 1/\sqrt{1-\beta^2}$. K_0, K_1, I_0 and I_1 are modified Bessel functions of the second and the first kind, respectively and of the zeroth and first order, correspondingly. J_0 and J_1 are Bessel functions of the first kind, and the zeroth and first order, correspondingly. ν_n are defined by equation $J_0(\nu_n) = 0$ and are understood to be ordered: $\nu_1 < \nu_2 < \ldots < \nu_n < \nu_{n+1} \ldots, n = 1, 2, \ldots\infty$. The sign of the imaginary parts of the propagation constants $\lambda_{bn} = \sqrt{k^2 - \nu_n^2/b^2}$ and $\lambda_{an} = \sqrt{k^2 - \nu_n^2/a^2}$ should be chosen positive (that choice is defined by the radiation condition): $Im\lambda_{dn} > 0$. In all equations above, $B_n^\pm$ are unknown coefficients to be defined by the boundary and continuity conditions in the plane $z = 0$.

Each term in the expressions for the diffracted field describes either a nth wave propagating in the positive z direction, if $k > \nu_n/a$, or an evanescent wave, if $k < \nu_n/a$.

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Similarly, each term in the expressions for the reflected field describes either a nth wave propagating in the negative z direction, if $k > \nu_n/b$, or an evanescent wave, if $k < \nu_n/b$. For any given k there are finite number of the propagating and infinite number of evanescent waves.

On a perfectly conducting wall the tangential component of the electric field should be zero. In our case that means that the radial component of electric field for all $b < r < a$ should be zero at $z = 0$. For the charge entering the narrow pipe

$$\widetilde{E}_r^-(r,0) = 0 \,. \tag{10}$$

Conversely, for the charge exiting the narrow pipe $\widetilde{E}_r^+(r,0) = 0$.

In the plane $z = 0$ but in the opening of the pipe all the fields components should be continuous. Hence for $0 < r < b$

$$\widetilde{E}_r^+(r,0) = \widetilde{E}_r^-(r,0), \ \widetilde{E}_z^+(r,0) = \widetilde{E}_z^-(r,0) \tag{11}$$

Since three functions E_r, E_z and H_θ are solutions of Maxwell equations, the continuity condition for H_θ is fulfilled as soon as conditions (11) are satisfied. Substituting expressions (4),(5) and (7),(8) into Eqs. (10),(11) we obtain the system of transcendental equations which define unknown coefficients $B_n^\pm$.

Using orthogonality of Bessel functions it is possible to transfer the system of transcendent equations into a linear system of algebraic equations. We will do this for the most interesting ultrarelativistic case $\gamma \to \infty$.

Introduce dimensionless coefficients $g^\pm$ by the following expressions: $B_m^\pm = (2iqa/\pi c)g_m^\pm$. Excluding g_n^+ we obtain system of equations for g_n^- only:

$$\Sigma_m[T_{lm} + \delta_{lm}\widetilde{\lambda}_{al}J_1^2(\nu_l)]g_m^- = F_l \quad , \tag{12}$$

where $p = b/a$,

$$F_l = J_0(\nu_l p)/\nu_l^2 \,, \tag{13}$$

$$T_{lm} = 4\nu_m^2 p^3 J_0(\nu_m p)J_0(\nu_l p)\Sigma_n \frac{\widetilde{\lambda}_{bn}}{(\nu_n^2 - \nu_m^2 p^2)(\nu_n^2 - \nu_l^2 p^2)} \,, \tag{14}$$

and $\widetilde{\lambda}_{al} = a\lambda_{al}, \widetilde{\lambda}_{bl} = b\lambda_{bl}$.

Notice that both diffracted and reflected fields are limited for $\gamma \to \infty$ since $B_n^\pm$ do not depend on energy in this limit.

Up to now all the relations are exact. Since we have no way to solve the infinite system (12) exactly, its approximate numerical solution is used. The system is truncated to a finite size and coefficients $g_m^\pm$ are found by matrix inversion. The approximate expressions for the electromagnetic field components are then obtained using truncated Eqs. (4-9). Figures 2–7 illustrate how the boundary and continuity conditions are satisfied by the approximate solutions for an intermediate value $p = 0.3$ with the matrix size 20x20. The discontinuity of the E_r at the sharp corner of the boundary can never be approximated by any finite numbers of eigenfunctions. Nevertheless in all other regions the solution seems to be quite satisfactory.

The longitudinal coupling impedance now can be obtained by integrating the synchronous component of E_z at $r = 0$ (*i.e.*, along the particle path).[3] The synchronous term of E_z^S gives the following expression for the impedance per unit length:

$$\frac{dZ^S(k)}{dz} = \frac{2ik}{c\gamma^2\beta^2} \ln\frac{a}{b} \tag{15}$$

This term goes to zero in the ultrarelativistic limit. From the radiation part of the field E_z^R after performing the integration and some algebra we get:

$$Z_{in}(k) = -\frac{Z_0}{\pi p}\{\Sigma_n g_n^+[\kappa p + (\kappa^2 p^2 - \nu_n^2)^{\frac{1}{2}}] - p\Sigma_n g_n^-[\kappa - (\kappa^2 - \nu_n^2)^{\frac{1}{2}}]\}. \tag{16}$$

where $4\pi/c = Z_0 = 377$ Ohm is the impedance of the free space, and the normalized frequency $\kappa = a\omega/c$.

The impedance for the case of a charge exiting the narrow pipe can be found from a similar formula:

$$Z_{out}(k) = -\frac{Z_0}{\pi p}\{\Sigma_n g_n^+[\kappa p - (\kappa^2 p^2 - \nu_n^2)^{\frac{1}{2}}] - p\Sigma_n g_n^-[\kappa + (\kappa^2 - \nu_n^2)^{\frac{1}{2}}]\}. \tag{17}$$

Figures 8 and 9 present the real and the imaginary parts of the longitudinal impedance, respectively, for incoming and outgoing charge for $p = 0.3$ as functions of κ. The resonance behavior of the impedance is clearly exhibited.

An extended version of this work can be found in SLAC–PUB–3965. Discussions with many people were helpful while preparing this paper. We are grateful to all of them.

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2. J. A. Stratton, "Electromagnetic Theory," N.Y., McGraw-Hill, 1941.
3. S. Kheifets, L. Palumbo and V. G. Vaccarro, "Longitudinal Impedance of a Semi-Infinite Circular Waveguide," CERN LEP-TH/85-23 (June 1985).

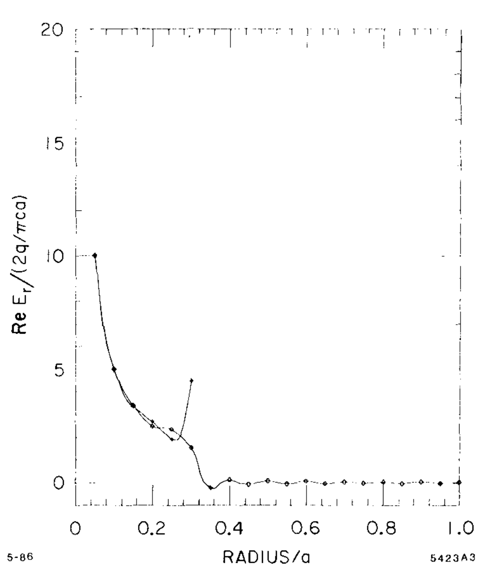

Fig. 2. Example of the continuity and boundary conditions for the real part of the radial electrical field component.

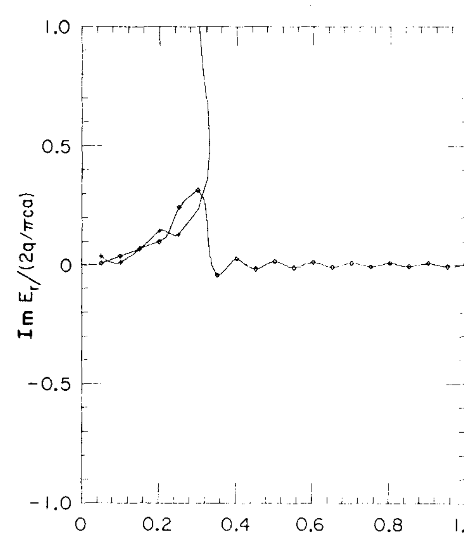

Fig. 3. The same as in Fig. 2 except for the imaginary part of the radial electric field component.

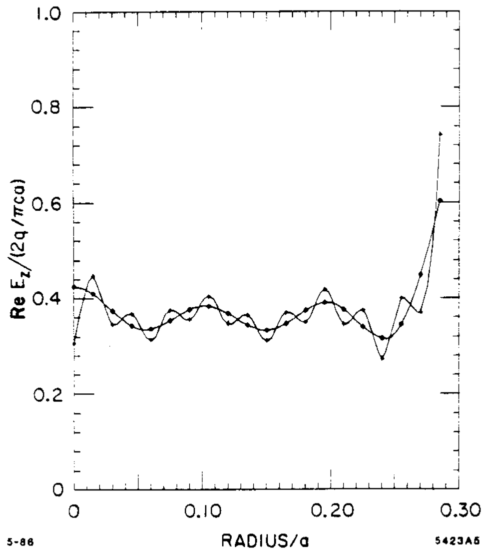

Fig. 4. The same as in Fig. 2 except for the real part of the longitudinal electric field component.

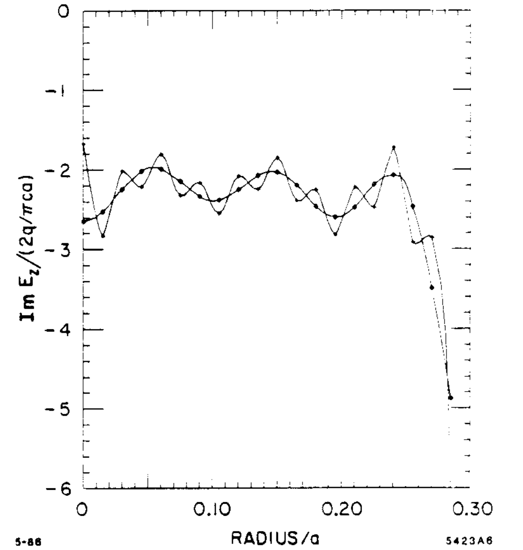

Fig. 5. The same as in Fig. 2 except for the imaginary part of the longitudinal electric field component.

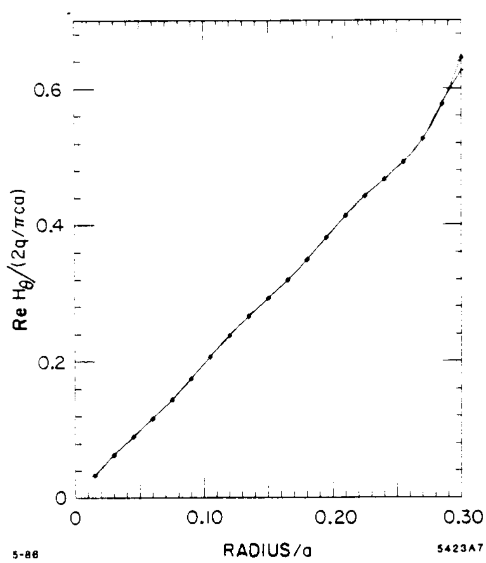

Fig. 6. The same as in Fig. 2 except for the real part of the azimuthal magnetic field component.

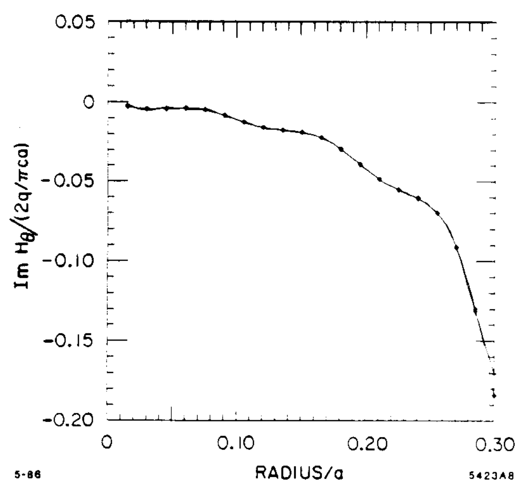

Fig. 7. The same as in Fig. 2 except for the imaginary part of the azimuthal magnetic field component.

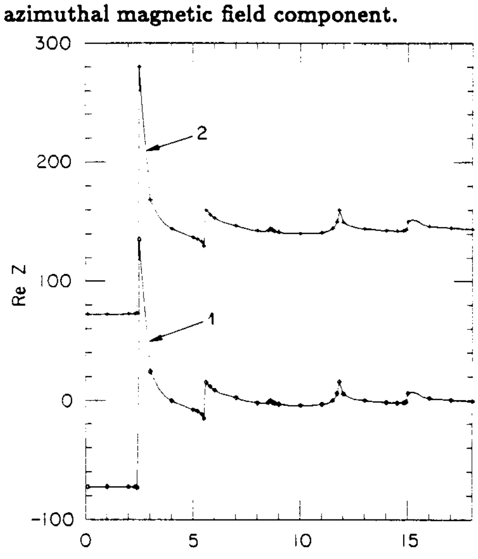

Fig. 8. The real part of the longitudinal coupling impedance as function of frequency ($\kappa = a\omega/c$, 1- Z_{in}, 2- Z_{out}).

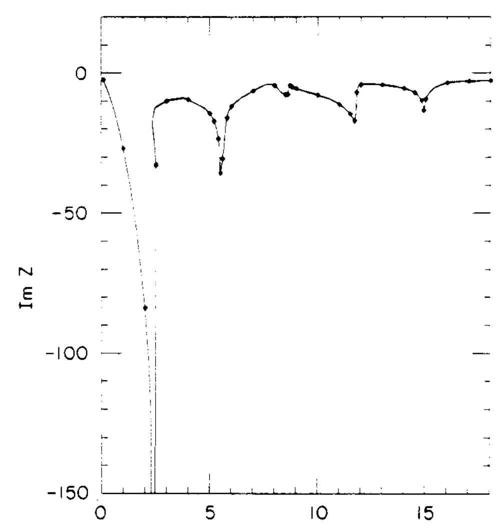

Fig. 9. The imaginary part of the longitudinal coupling impedance as a function of frequency ($\kappa = a\omega/c$, Z_{in} and Z_{out} are on the same curve).