TRANSIENTS IN ELECTRON LINACS

B. Epsztein and J.P. Mangin
C. S. F.
91-Orsay, France

Introduction

A number of physics studies require short pulses and high peak currents from electron linacs. The duration of these pulses can be much shorter than the filling time of the accelerator, in which case the operation cannot reach a steady state. Hence, it is necessary to develop a theory valid for such conditions. Leiss and Schrack[1], in particular have made an exhaustive analysis of this case, later exposed in abridged form by Leiss during the 1965 Linear Accelerator Conference. The main result of this study was a large increase of the beam-loading during the initial period of the beam pulse (about ten r.f. cycles) with respect to what would be computed in a conventional manner, disregarding the dispersion of the wave guide.

The present study was initiated with two main objects in mind : First, to get a better physical understanding of the phenomenon and its causes since the mathematical intricacy of Leiss' analysis did not allow an intuitive appraisal of the situation. Second, to obtain approximate analytic expressions for the beam loading in order to avoid the use of computers and be able to appreciate rapidly the influence of each parameter.

The unexpected result was that our analysis did not show any trace of the extra beam loading discovered by Leiss, in spite of the fact that most initial assumptions were identical for both cases. Instead, our result is very close to the one obtained in the more naive approach neglecting dispersion effects.

We shall now give the detail of this analysis, followed by a discussion on the reasons for the discrepancy between Leiss and us.

Analysis of Transient Effects

Assumptions

As already mentionned, most assumptions are essentially the same as in Leiss :

1) The wave guiding structure is assumed to be periodic and to carry a single mode. It is, however, possible to take into account higher order modes considering each one separately and adding their contribution.
2) The beam is supposed to be made up of very tight bunches which can be assimilated to delta functions of time and space. These are further assumed to experience no change in velocity due to the fields in the structure. This last assumption is fulfilled in the case of ultra relativistic beams. Hence, the electrons will be supposed to propagate at the speed of light.
3) The gaps of the different cells are supposed to be well separated in space, with the consequence that a single bunch crossing a gap induces fields in this gap only, not in the neighbouring ones. The effect of the transit time through the gap will be assumed negligible or else taken into account by a constant factor. In the analysis, this factor will be omitted for the sake of simplicity.
4) The dispersion of the wave guiding structure will be the same as in Leiss, though written somewhat differently.
5) Unless otherwise specified, the structure is supposed to be lossless.

Notation

φ phaseshift between adjacent cells at frequency ω
ω_o midband angular frequency corresponding to $\varphi = \pi/2$
$\Delta\omega$ half angular bandwidth of the structure
n cell number
p cell length
$v_g = p\Delta\omega$ group velocity at midband ($\omega = \omega_o$)
q charge of an electron bunch
C equivalent capacitance in a single cell
$L = 1/C\omega_o^2$ equivalent inductance in a single cell
$\mathcal{L} = L/p$ inductance per unit length
$\tau = p/c$ transit time of a bunch across one cell.

Case of a Single Bunch

Excitation of a Single Cell. We shall first consider the excitation of the structure by a single bunch. By adding the field contribution of the successive bunches it is then possible to obtain the field configuration as a function of time and cell number as well as the beam energy loss.

As a very first step, let us compute the excitation of a single cell by a single bunch. We shall admit that the section can be considered as a set of coupled resonators. Let us suppose that an electron bunch appears suddenly at the beginning of a cell, crosses the cell and then disappears. If the transit time through the gap is very small, or if the coupling device between cells is far enough removed from the gap (case of coupling by loops or inductive holes) so that no r.f. field has time to reach it during the transit of the bunch through the gap, the excited cell can be at that time considered as uncoupled to its neighbours. Consequently, the initial field in a single cell is the same as for an uncoupled resonator. This assumption is rather well satisfied in practice, especially for narrow bandwidth structures.

The voltage appearing across the gap of a resonator after the passage of a single bunch is, in case of negligible transit time :

$$V = q/C \qquad (1)$$

This rather obvious result can be refined by taking into account the finite length and radius of the gap through a development of the current delta function into a Fourier integral, but apart from a change in the numerical value of C, the relation (1) remains valid.

Coming back to the entire waveguide, we have the following situation at time t = 0, right after the crossing of the gap of a single cell which will be called the n° 0 cavity, the cells on the right having a positive number, those on the left a negative one : all cavities are empty, except for cavity n° 0 across the gap of which a voltage given by equation (1) is developed.

In order to proceed further, we have to define the dispersion of the structure. We shall choose a dispersion equation which, though different in shape, is in fact strictly equivalent to the one used in Leiss. This is :

$$\omega = \omega_o - \Delta\omega \cos\varphi \qquad (2)$$

It fits measured dispersions to an excellent approximation. Thus the general expression of the voltage developed across the n^{th} gap when the structure is excited at a single frequency ω can be written :

$$U_n(t) = U_o\, e^{j(\omega t - n\varphi)} = U_o\, e^{j[(\omega_o - \Delta\omega\cos\varphi)t - n\varphi]} \qquad (3)$$

which at time t = 0 reduces to :

$$U_n(o) = U_o\, e^{-jn\varphi} \qquad (4)$$

We are now faced with the problem of satisfying the initial conditions, that is, no field anywhere except in the n° 0 cell, by means of a sum of terms like those in equation (4) which are the only ones to satisfy the dispersion equation. This task is readily accomplished by writing :

$$G(n,o) = \frac{q}{2\pi C}\int_{-\pi}^{\pi} e^{-jn\varphi}\, d\varphi \qquad (5)$$

where it is easy to see that $G(n,o)$ is zero for any n except $n = 0$, for which $G(o,o) = q/C$.

The subsequent behavior of the field will be expressed by :

$$G(n,t) = \frac{q}{2\pi C}\int_{-\pi}^{\pi} e^{j(\omega t - n\varphi)}\, d\varphi$$

$$= \frac{q}{2\pi C}\int_{-\pi}^{\pi} e^{j\omega_o t}\, e^{-j(\Delta\omega t\cos\varphi + n\varphi)}\, d\varphi$$

$$= \frac{q}{C}\, e^{j\left(\omega_o t - \frac{n\pi}{2}\right)} J_n(\Delta\omega t)$$

J_n being the Bessel function of first kind and n^{th} order one can immediately check that this expression

1) satisfies the initial conditions
2) is made up of waves obeying the dispersion relation hence that it gives the response of the section for the excitation of a single cell by a single bunch.

The physical meaning is clear : it is a well known property of Bessel functions of first kind that their magnitude is negligibly small until the argument reaches approximately the value n . So, a field starts to appear when

$$\Delta\omega t \simeq n \qquad (7)$$

or, since $\Delta\omega = \frac{v_g}{p}$

$$v_g t \simeq np \qquad (8)$$

which means that the energy propagates with approximately the velocity v_g , group velocity at the midband frequency, or still, largest group velocity in the band. The propagation is symmetrical with respect to the excited cell. It is to be mentionned that equation (6) has (, physical meaning for $t \geq 0$. For $t < 0$ it must be assumed that the field is zero everywhere, so that the complete solution is the product of the right hand side of equation (6) by a step function starting at t = 0.

Case of a Lossy Structure. We shall assume that the current and stored energy distribution in the structure do not vary appreciably across the passband. In that case, we can consider the Q of the section as constant. The presence of losses implies a small imaginary component in the frequency, the term $\omega\left(1 - \frac{j}{2Q}\right)$ replacing ω written for the lossless case. Hence the dispersion formula

$$\omega\left(1 - \frac{j}{2Q}\right) = \omega_o - \Delta\omega\cos\varphi \qquad (9)$$

Calling α the attenuation per unit length, one can check that the usual relation :

$$\alpha = \frac{\omega}{2Q v_g}$$ is fulfilled.

Introduction of (9) into (6) results, in the first approximation, in the multiplication of its right hand side by a factor $e^{-\frac{\omega_o t}{2Q}}$

From this point on, we shall again assume the structure lossless.

Excitation of a Section by a Single Bunch

We are now in a position, by summing the contribution of each cell, to compute the field excited all along the structure by a single bunch. For the sake of convenience, we shall give the number zero to the cavity where the total field is computed, the bunch being supposed to cross this cavity at time t = 0. The problem is formally solved by adding the contribution of the successive cavities, taking into account the delay between their excitation due to the finite velocity of the bunch :

$$V_o(t) = \sum_{-m}^{n=\frac{t}{\tau}} G(-n, t - n\tau) \qquad (10)$$

or $V_o(t) = \sum_{-m}^{n=\frac{t}{\tau}} G(n, t - n\tau)$ since $G(-n,t) = G(n,t)$

$$V_o(t) = \frac{q}{C}\, e^{j\omega_o t} \sum_{-m}^{n=\frac{t}{\tau}} e^{-jn\left(\omega_o\tau + \frac{\pi}{2}\right)} J_n(\Delta\omega t - n\Delta\omega\tau) \qquad (11)$$

where $-m$ is the cell number at the beginning of the section and τ the transit time of the bunch across one cell. We have not been able to achieve the summation of this expression in closed form and the use of a computer would be required to get exact numerical values. It is possible, however, to make approximations which allow analytical summation and give formulas readily interpretable from a physical point of view. (2)

First approximation. The method consists in develo-

ping the Bessel functions in a Taylor series around $\Delta\omega t$

$$J_n(\Delta\omega t - n\Delta\omega\tau) = J_n(\Delta\omega t) - n\Delta\omega\tau J_n'(\Delta\omega t) + \frac{n^2\Delta\omega^2\tau^2}{2} J_n''(\Delta\omega t) - \cdots$$

This development, limited to its first terms, is in particular valid as long as $n\Delta\omega\tau \ll 1$. Remembering that $\Delta\omega = v_g/p$ and $\tau = p/c$ we get the condition :

$$n \ll c/v_g \qquad (13)$$

Now, the power released in the n^{th} cavity requires a time : np/v_g to reach the 0^{th} cavity. Neglecting the bunch transit time with respect to the r.f. energy transit time, it is obvious that the influence of the + n^{th} and - n^{th} cavities will be felt on the 0^{th} cavity at time $t = np/v_g$, at which time the influence of all higher rank cavities is negligible. Hence, condition (13) is certainly satisfied provided

$$t \ll \frac{cp}{v_g^2} \qquad (14)$$

This condition, applied in the common case of 4 cells per wavelength can be expressed as :

$$\omega_o t \ll \frac{\pi}{2}\left(\frac{c}{v_g}\right)^2$$

To get an order of magnitude we shall suppose $c/v_g = 50$ which is a value often found in practice. Then the condition (14) is satisfied when the number of r.f. cycles remains less than about one hundred. Fortunately, this is the period of interest to us, when the phenomenon of extra beam loading is supposed to happen.

One more assumption is to be made, that the 0^{th} cell is not too close to the section input, that is $m > 10$ for instance. With all these restrictions, the field in the 0^{th} cavity is the same as if the section were infinite and equation (11) can be replaced by :

$$V_o(t) = \frac{q}{C} e^{j\omega_o t} \sum_{-\infty}^{\infty} e^{-jn(\omega_o\tau + \frac{\pi}{2})} \left[J_n(\Delta\omega t) - n\Delta\omega\tau J_n'(\Delta\omega t) + \cdots \right] \qquad (15)$$

The summation taking into account only the first of the terms between brackets is readily accomplished by means of the Jacobi-Anger formula, which is the basic formula for spectrum analysis of frequency modulation :

$$\sum_1 = \sum_{-\infty}^{\infty} e^{-jn(\omega_o\tau + \frac{\pi}{2})} J_n(\Delta\omega t) = e^{-j\Delta\omega t \sin(\omega_o\tau + \frac{\pi}{2})} \qquad (16)$$

and since $\omega_o\tau \simeq \omega\tau = \varphi$ (17)

φ being the synchronous phase

$$\sum_1 = e^{-j\Delta\omega t \cos\varphi} \qquad (18)$$

Hence, with the help of (2)

$$V_{o1}(t) = \frac{q}{C} e^{j\omega t} \qquad 0 < \omega t < 200\pi \qquad (19)$$

V_{o1} being the first approximation to the voltage developed in cavity 0. It should be noticed that this approximation corresponds to the simultaneous excitation of all cavities with correct phases at time t = 0. The striking result, in this approximation, is that the cavity starts right at the beginning to oscillate at the synchronous frequency ω, without any phase or amplitude modulation, in perfect agreement with the simple calculations neglecting dispersion.

Second approximation. The second term of expansion (15) can be summed as follows :

$$\sum_2 = \sum_{-\infty}^{\infty} e^{-jn(\omega_o\tau + \frac{\pi}{2})} n\Delta\omega\tau J_n'(\Delta\omega t) \qquad (20)$$

$$= \sum_{-\infty}^{\infty} e^{-jn(\omega_o\tau + \frac{\pi}{2})} \frac{n\Delta\omega\tau}{2} \left(J_{n-1}(\Delta\omega\tau) - J_{n+1}(\Delta\omega\tau) \right)$$

putting $\omega_o\tau + \frac{\pi}{2} = u$ (21)

$$\sum_2 = \frac{\Delta\omega\tau}{2} \sum_{-\infty}^{\infty} e^{-jnu} n\left(J_{n-1} - J_{n+1}\right)$$

$$= j\frac{\Delta\omega\tau}{2} \frac{d}{du} \sum_{-\infty}^{\infty} e^{-jnu} \left(J_{n-1} - J_{n+1}\right)$$

$$= j\frac{\Delta\omega\tau}{2} \frac{d}{du} \left[e^{-ju} \sum_{-\infty}^{\infty} e^{-jnu} J_n - e^{ju} \sum_{-\infty}^{\infty} e^{-jnu} J_n \right]$$

$$= \frac{\Delta\omega\tau}{2} \frac{d}{du} \left[2\sin u \sum_{-\infty}^{\infty} e^{-jnu} J_n \right]$$

Using again Jacobi-Anger

$$\sum_2 = \frac{\Delta\omega\tau}{2} \frac{d}{du} \left[2\sin u \; e^{-j\Delta\omega t \sin u} \right]$$

$$= \frac{\Delta\omega\tau}{2} \left(2\cos u - 2j\Delta\omega t \sin u \cos u \right) e^{-j\Delta\omega t \sin u} \qquad (22)$$

Hence, combining with (17) and (21)

$$\sum_2 = \frac{\Delta\omega\tau}{2} \left(-2\sin\varphi + 2j\Delta\omega t \sin\varphi\cos\varphi \right) e^{-j\Delta\omega t\cos\varphi} \qquad (23)$$

Now, we cannot be satisfied with the approximation (17) made to obtain the first order term. We have to improve it to include terms of the order we are looking for. So (17) becomes, taking (2) into account

$$\omega_o\tau = \varphi + \Delta\omega\tau\cos\varphi$$

and

$$V_{o1}(t) = \frac{q}{C} e^{j\left[\omega_o t - \Delta\omega t \cos(\varphi + \Delta\omega\tau\cos\varphi)\right]}$$

Since $\Delta\omega\tau$ is very small, the exponent can be developed to first order as :

$$\omega_o t - \Delta\omega t\cos\varphi + \Delta\omega t \sin\varphi\, \Delta\omega\tau\cos\varphi = \omega t + \Delta\omega t\, \frac{\Delta\omega\tau}{2}\sin 2\varphi$$

so that V_{o_1} becomes, to this approximation

$$V_{o_1}(t) = \frac{q}{C}\, e^{j\omega t}\left(1 + j\,\Delta\omega t\, \frac{\Delta\omega\tau}{2}\sin 2\varphi\right)$$

combining with (23) we obtain the second approximation

$$V_{o_2}(t) = \frac{q}{C}\, e^{j\omega t}\left[1 + \Delta\omega\tau\sin\varphi\right]$$

and, from the definition of $\Delta\omega$ and τ

$$V_{o_2}(t) = \frac{q}{C}\, e^{j\omega t}\left[1 + \frac{v_g}{c}\sin\varphi\right] \quad (0 < \omega t < 200\pi) \qquad (24)$$

One should remember that, from the definition of v_g (group velocity at midband) $v_g \sin\varphi$ represents the group velocity at the synchronous frequency.

Equation (24) shows a negligible change with respect to (19). The corrective term does not introduce any phase shift and is small enough to be neglected in practice. As a result no extra beam loading is to be expected. This is shown by a computation of the average field induced by the bunch which is obtained by summing the voltage drops over a unit length :

Since $C = 1/L\omega^2$ and $L = \mathcal{L}p$, the average field seen by the beam is given by :

$$E_b = \mathcal{L}\omega^2 q$$

which corresponds exactly to the classical formula. The difference by a factor 2 between both formulas is due to our definition of $\mathcal{L}$ since the stored energy per unit length would be written in our notation as :

$W = E^2/2\,\mathcal{L}\omega^2$ instead of the usual $W = E^2/\mathcal{L}\omega^2$

Comparison with Leiss Theory

Though both theories start with similar assumptions, their results are by no means compatible. The techniques used in both cases are different :whereas Leiss uses an expansion of the fields in terms of time and frequency as conjugate variables (in his Laplace transform), we use phaseshift and cavity number for the same purpose, which is equivalent to the use of space coordinates and propagation constants as conjugate variables. Both techniques are perfectly legitimate and equivalent, as evidenced by a number of examples given by Stratton in "Electromagnetic theory". The choice between them is simply a matter of convenience. Still, the first discrepancy arises almost at the beginning, for the expression of $G(n, t)$. In our notation, Leiss obtains :

$$G_L(n,t) = \frac{n J_n(\Delta\omega t)}{\Delta\omega t}\, e^{j\left(\omega_o t - \frac{n\pi}{2}\right)} \qquad (25)$$

(The factor $\Delta\omega$ in the denominator has been added in order to get a dimensionless expression) whereas we have :

$$G(n,t) = J_n(\Delta\omega t)\, e^{j\left(\omega_o t - \frac{n\pi}{2}\right)} \qquad (26)$$

This, as we recall, is the voltage induced in the waveguide by a single bunch passing through a single cavity. Let us examine what happens in the 0^{th} cavity, which is the cavity directly excited by the bunch. In our theory, the amplitude of the voltage across the gap is $J_0(\Delta\omega t)$; in Leiss theory, it is strictly zero all the time. This rather surprising result is explained by the fact that it has been explicitly assumed by Leiss : "As the current pulse passes point u (this is our cavity n° 0) it induces a delta function voltage and current impulse in the waveguide ....".[3] This statement cannot be correct, since it gives the value of the voltage at this point at any time which, in fact, is part of the problem to be solved. In our opinion, the only correct statement of the initial condition is expressed by the fact that at time t = 0 some voltage is induced on the 0^{th} cavity, all other cavities being empty. This way, no assumption is being made on the further behavior of the field anywhere in the waveguide and the fact that we have been able to carry out our calculation without further assumption shows that it is sufficient.

Going back to the Leiss expression (25) and using a well known identy, this becomes

$$G_L(n,t) = \frac{1}{2}\left[J_{n-1}(\Delta\omega t) + J_{n+1}(\Delta\omega t)\right] e^{j\left(\omega_o t - \frac{n\pi}{2}\right)}$$

which can also be written as :

$$G_L(n,t) = e^{-j\frac{\pi}{2}}$$

$$\left[J_{n-1}(\Delta\omega t)\, e^{j\left[\omega_o t - (n-1)\frac{\pi}{2}\right]} - J_{n+1}(\Delta\omega t)\, e^{j\left[\omega_o t - (n+1)\frac{\pi}{2}\right]}\right]$$

This, within the unimportant phase factor $e^{-j\frac{\pi}{2}}$ represents in our theory the field excited at time t = 0 by two half bunches, one made of electrons passing through cavity + 1, the other of opposite charge passing through cavity - 1. It is not surprising, because of antisymmetry, that the field remains zero in the 0^{th} cavity.

This difference alone, however, is not sufficient to explain the large beam loading found by Leiss. If we consider for example the case of a $\pi/2$ phase shift, the effect of two half bunches of opposite sign half a wavelength apart should be almost identical to the effect of a single bunch.

The explanation, we think, lies in the fact that, while summing up in one cavity the fields originating from the excitation of the other cavities, Leiss systematically neglects the contribution of the field sources located downstream with respect to this cavity, taking only into account what he calls "forward waves". While this procedure could perhaps be justified in the case of a steady state, forward wave interaction, it is certainly unwarranted for a transient state. In that case, the so called "backward waves" (with negative group velocity) bring an important contribution to the total field

of the cavity, in fact so important as to cancel the spurious beam loading due to the "forward waves" alone. As far as electrons are concerned, they do not care whether the energy comes from right or left, as evidenced by backward wave interaction; they are acted upon by the total field.

In conclusion, we have outlined a theory for transients in linacs. In spite of some approximations (for example the dispersion equation is not a solution of Maxwell's equations since it should be at least even in ω) this theory can still be accurate enough to satisfy most practical purposes. It can of course be extended to problems which were not considered in this paper such as the filling up of a linac section, the transient interaction with higher order modes, etc ....

Acknowledgement

We would like to thank professor Lapostolle for a very helpful and stimulating discussion.

(1) J.E. Leiss and R.A. Schrack "Transient and beam loading phenomena in linear electron accelerators",
N.B.S. internal report Oct. 62
This report will be referred for brevity as Leiss.
(2) It is to be mentionned that this method takes automatically all space harmonics into account since it gives the total field at each gap.
(3) See ref. (1) page 27.

Note written in proof : We have recently succeeded in obtaining the complete summation of equation (15) in closed form without any approximation. The result is

$$V_o(t) = \frac{q}{C} \frac{e^{j\omega t}}{1 - \frac{v_g}{c} \sin\varphi}$$

which is very close to the result given by equation (24). The conclusions remain obviously the same.