# SIMPLE MODELS FOR BEAM-BLOWUP†

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## I. Introduction and Summary

The study of beam-blowup commenced a fairly large number of years ago, and many of the results of this paper are already contained in the considerable existing literature about it.[1] Much of the following discussion and exposition is included principally for the sake of completeness and a unified point of view.

A chain of independent cylindrical cavities is a simple model for a linac and is the one which we consider in Section III. The normal-mode fields are described in detail, as is their interaction with an off-axis beam, and the resulting equation for beam deflection is displayed. The magnitude of the coefficient in this equation has a simple physical interpretation; namely, it is the electrostatic force attracting the bunch to its image induced in the wall of the accelerating tube.

Section IV is devoted to a discussion of deflection caused by coöperative buildup of the transverse fields by the beam bunches in a more general linac model. An equivalent circuit description of a chain of cavities with linear coupling to nearest neighbors is worked out. The relations between the eigenvectors and eigenvalues of this system and the parameters of the bunched off-axis beam which are relevant to beam-blowup are discussed, as is the section-to-section variation in the normal modes which is consistent with coherent transverse-field buildup.

The next few paragraphs contain an elementary discussion of energy flow between the various degrees of freedom in a linac.

## II. Degrees of Freedom in the Linac

The purpose of a linear accelerator is to transform electrical energy into unidirectional kinetic energy of a beam of charged particles. Figure 1 is a block diagram showing how this is done. The RF source is the circle in the upper left, the box at the upper right represents the longitudinal kinetic energy in the beam. The presence of the other elements of Fig. 1, which are modes of storage, transfer, and dissipation of energy, is regrettable but unavoidable.

For instance a TM mode in the accelerator cavity serves as an intermediary between the RF power supply and the longitudinal kinetic energy of the beam. This accelerating mode interacts with the charge-current of the beam bunches. McMillan's application[2] of Earnshaw's Theorem to the linac tells us, though, that any interaction (top center circle on Fig. 1) which causes a phase-stable acceleration necessarily also causes defocussing. Hence we must add another leg (lower left) to the diagram, ending in a box containing transverse kinetic energy. The defocussing can then be neutralized by inserting strong-focussing quadrupole lenses, which can exchange energy between the transverse and longitudinal kinetic energy boxes.

This completes the block diagram for an ideal accelerator whose structure has no deflecting modes. This is, unfortunately, impossible, and the part of Fig. 1 below the diagonal, that involving the deflecting modes, must be added. Energy is fed into these modes mainly from the longitudinal kinetic energy of the off-axis beam, and from them goes into the transverse kinetic energy, the box at the lower left. Too much transverse kinetic energy can be disastrous. The aim of studying beam-blowup is to find means for keeping the amount small.

The deflecting mode amplitude in a given cavity is built up over the course of time by the coherent contributions of each passing bunch. This is the feature which puts its effect on the transverse beam motion in a different category from that of the accelerating mode, which is essentially known, constant in time and compensated for by the quads.

Energy flow around the lower right-hand corner of Fig. 1, from longitudinal to transverse kinetic energy, can be inhibited in three distinct ways. First, the structure might be so designed that it is very lossy for these modes but not for the accelerating mode. Second, the phase of the deflecting mode in some of the sections can be reversed and its magnitude increased enough so that the net deflection over many sections averages to zero. Third, the properties (frequency, eigenvectors) of the deflecting modes in the various sections can be tampered with in such a way that the contributions of successive bunches to the deflecting mode amplitude become incoherent.

In practice, it is impossible to consider calculating all of the interactions in Fig. 1 at once. Most calculations take account of only one or two of the connections, and, in fact, consider some of the energy flows to be unidirectional.

We will truncate the diagram as far as is reasonable for our purposes. We keep the two kinetic energy boxes and the deflecting mode energy box and the interactions connecting them. Furthermore, we consider the longitudinal velocities to be prescribed functions of time.

The discussions of the next two sections differ only in the models they assume for the deflecting mode.

## III. Deflecting Mode Buildup in a Chain of Independent Cylindrical Cavities.

The normal modes of the electromagnetic fields and their interaction with the beam can be described in complete detail for the cylindrical cavity with no holes or drift tubes. For numerical orientation and because it will lead to some physical insight, we do so now.

### A. Beam Current as the Source of the Field

The vector potential $\vec{A}(r,t)$ is the solution of

$$\Box^2 \vec{A} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_o \vec{j} \qquad (1)$$

(in which $\vec{j}\,(\vec{r},t)$ is the beam current) which satisfies metallic boundary conditions on the conducting surfaces;

$$B_\perp = E_{||} = 0. \qquad (2)$$

$\vec{B}$ and $\vec{E}$ are the fields derivable from $\vec{A}$ by

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\partial \vec{E}}{\partial t} = c^2\, \vec{\nabla} \times \vec{B}. \qquad (3)$$

A formal solution of Eq. (1) is

$$\vec{A}(\vec{r},t) = -\mu_o \int G(\vec{r},\vec{r}';t,t')\vec{j}(\vec{r}',t')d^3r'dt', \qquad (4)$$

where the Green's function satisfies

$$\Box^2 G(\vec{r},\vec{r}';t,t') = \delta(\vec{r}-\vec{r}')\,\delta(t-t'), \qquad (5)$$

with metallic boundary conditions. A Fourier integral representation

$$G(\vec{r},\vec{r}';t,t') = \frac{1}{2\pi}\int_C \widetilde{G}(\vec{r},\vec{r}';\omega)e^{i\omega(t-t')}d\omega \qquad (6)$$

defines $\widetilde{G}$, which satisfies

$$\left[\nabla^2 + (\omega/c)^2\right]\widetilde{G}(\vec{r},\vec{r}';\omega) = \delta(\vec{r}-\vec{r}'). \qquad (7)$$

The contour C in Eq. (6) will be chosen so as to make G a causal function; i.e., it will vanish for $t < t'$.

### B. Normal Modes of the Cavity

A set of scalar functions $\mathcal{G}_\omega(\vec{r})$ satisfying

$$\left(\nabla^2 + (\omega/c)^2\right)\mathcal{G}_\omega(\vec{r}) = 0 \qquad (8)$$

facilitates an explicit expression for $\widetilde{G}$, namely,

$$\widetilde{G}(\vec{r},\vec{r}';\omega) = \sum_{\omega'} \frac{\mathcal{G}_{\omega'}(\vec{r})\,\mathcal{G}^*_{\omega'}(\vec{r}')}{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega'}{c}\right)^2}, \qquad (9)$$

provided that

$$\sum_\omega \mathcal{G}_\omega(\vec{r})\,\mathcal{G}^*_\omega(\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$\int d^3r\, \mathcal{G}^*_\omega(\vec{r})\,\mathcal{G}_{\omega'}(\vec{r}) = \delta_{\omega,\omega'}, \qquad (10)$$

and that $\mathcal{G}$ satisfies appropriate boundary conditions. The transverse components of $\vec{j}$ can be neglected for our purposes; therefore $A = (0,0,A_z)$ and we require

$$\frac{\partial \mathcal{G}_\omega}{\partial \varphi} = 0 \quad \text{at } r = a$$

$$\frac{\partial^2 \mathcal{G}_\omega}{\partial r \partial z} = \frac{\partial^2 \mathcal{G}_\omega}{\partial \varphi \partial z} = 0 \quad \text{at } z = 0, \ell, \qquad (11)$$

where r, φ, and z are cylindrical coördinates in one cavity, and a and ℓ are the radius and length of a cavity, respectively (See Fig. 2).

A set of $\mathcal{G}$'s which complies with these specifications is

$$\mathcal{G}_\omega(\vec{r}) = A_{\mu k t}\cos\mu(\varphi-\varphi_o)\cos\frac{\pi k z}{\ell}\, J_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right), \qquad (12)$$

provided that

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{\pi k}{\ell}\right)^2 + \left(\frac{\pi\beta_{\mu t}}{a}\right)^2, \qquad (13)$$

$$A^2_{\mu k t} = \frac{4}{\pi \ell a^2} \frac{1}{J^2_{\mu+1}(\pi\beta_{\mu t})} \frac{1}{1+\delta_{k,o}} \frac{1}{1+\delta_{\mu,o}} , \quad (14)$$

$k, \mu = 0, 1, 2, \ldots$

$t = 1, 2, 3, \ldots,$

and $\pi\beta_{\mu t}$ is the $t^{th}$ root of the $\mu^{th}$ Bessel function of the first kind. $(J_\mu(\pi\beta_{\mu t}) = 0)$ $\varphi_o$ is arbitrary.

Explicitly, then, (compare Morse and Feshbach[3] p. 1263)

$$\tilde{G}(\vec{r},\vec{r}';\omega) = \frac{4}{\pi\ell a^2} \sum_{\mu k t} \frac{\cos\mu(\varphi-\varphi')\cos\frac{\pi k z}{\ell}\cos\frac{\pi k z'}{\ell}}{(1+\delta_{k,o})(1+\delta_{\mu,o})\, J^2_{\mu+1}(\pi\beta_{\mu t})}$$

$$\times \frac{J_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right) J_\mu\left(\frac{\pi\beta_{\mu t} r'}{a}\right)}{\left(\frac{\pi k}{\ell}\right)^2 + \left(\frac{\pi\beta_{\mu t}}{a}\right)^2 - \left(\frac{\omega}{c}\right)^2} . \quad (15)$$

The electric and magnetic fields derived from G according to Eq. (3) are

$$\mathcal{B}_r = -\frac{\mu}{r}\sin\mu(\varphi-\varphi_o)\cos\frac{\pi k z}{\ell} J_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right)$$

$$\mathcal{B}_\varphi = -\frac{\pi\beta_{\mu t}}{a}\cos\mu(\varphi-\varphi_o)\cos\frac{\pi k z}{\ell} J'_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right)$$

$$\mathcal{B}_z = 0 \quad (16)$$

$$\dot{\mathcal{E}}_r = -c^2\frac{\pi k}{\ell}\frac{\pi\beta_{\mu t}}{a}\cos\mu(\varphi-\varphi_o)\sin\frac{\pi k z}{\ell} J'_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right)$$

$$\dot{\mathcal{E}}_\varphi = \frac{c^2\mu}{r}\frac{\pi k}{\ell}\sin\mu(\varphi-\varphi_o)\sin\frac{\pi k z}{\ell} J_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right)$$

$$\dot{\mathcal{E}}_z = c^2\left(\frac{\pi\beta_{\mu t}}{a}\right)^2\cos\mu(\varphi-\varphi_o)\cos\frac{\pi k z}{\ell} J_\mu\left(\frac{\pi\beta_{\mu t} r}{a}\right) ,$$

where we have omitted the normalization factor $A_{\mu k t}$.

If we consider the effects of the deflection modes $\mu k t$ individually, then we can drop the sum in Eq. (15), and the fields induced by the current $\vec{j}$ will be proportional to those given in Eq. (16).

C. <u>The Fields Deflect the Beam</u>

Panofsky and Wenzel[4] have shown that the effective radial force exerted on a beam bunch during its traversal of a cavity is

$$\frac{d}{dt}(M\dot{r}_s) = e v_s \frac{\partial G_z(\vec{r},t)}{\partial r}\Bigg|_{\vec{r}=\vec{r}_s} \quad (17)$$

where e is the total charge of the bunch, $\vec{r}_s$ is its position, M its mass, $v_s$ its longitudinal velocity, and $\vec{G}$ is a vector potential which generates the electric field according to

$$\vec{E} = -\frac{\partial\vec{G}}{\partial t} . \quad (18)$$

Such a potential can be obtained from Eq. (4) by replacing the $\mathcal{G}_\omega'(\vec{r})$ in Eq. (9) with $-\mathcal{E}/(i\omega)^2$ and changing $\vec{j}$ to $j_z$. Then Eq. (17) becomes

$$\frac{d}{dt}(M\dot{r}_s) = \frac{e\mu_o v_s}{2\pi}\int_C d\omega \int d^3r' \int dt' \frac{e^{i\omega(t-t')}}{(i\omega)^2}$$

$$\times \frac{\frac{\partial\dot{\mathcal{E}}_z(\vec{r}_s)}{\partial r}\, \mathcal{G}^*_{\omega_T}(\vec{r}')}{\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_T}{c}\right)^2\right]} j_z(\vec{r}',t') . \quad (19)$$

This is the equation which determines the bunch deflection.

The correct causal behavior and asymptotic time-dependence of the Green's function (Eq. (5)) and the kernel in Eq. (19) is obtained by taking the contour C in both equations to be the real axis from $-\infty$ to $+\infty$, and displacing the poles at $\omega = \pm\omega_T$ into the upper half-plane to $\omega = \pm\omega_T + \frac{i\omega_T}{2Q}$. This causes G to vanish for $t < t'$, and makes it damp exponentially for $t \gg t'$.

If we now substitute the current of a perfectly bunched beam

$$j_z(\vec{r},t) = e\sum_{s'} v_{s'}\,\delta(\vec{r}-\vec{r}_{s'}) \quad (20)$$

into Eq. (19), and perform the $\omega$ and $r'$ integrals, we get

$$\frac{d}{dt}(M\dot{r}_s) = \frac{e^2 v_s(t)}{\omega_T^3\,\epsilon_o}\sum_{s'}\int_{-\infty}^{t} dt'\sin\omega_T(t-t')\, e^{-\frac{\omega_T(t-t')}{2Q}}$$

$$\times \frac{\partial\dot{\mathcal{E}}_z(\vec{r}_s)}{\partial r}\, \mathcal{G}^*_{\omega_T}(\vec{r}_{s'})\, v_{s'}(t') , \quad (21)$$

which, after using Eq. (16), becomes

$$\frac{d}{dt}(M\dot{r}_s) = \frac{4e^2c^2v_s(t)}{\pi\ell a^2\, J^2_{\mu+1}(\pi\beta)\,\epsilon_o}\;\frac{1}{1+\delta_{k,o}}\;\frac{1}{1+\delta_{\mu,o}}$$
$$\times\left(\frac{\pi\beta}{\omega_T a}\right)^3 \sum_{s'} \int_{-\infty}^{t} dt' \sin\omega_T(t-t')\, e^{\frac{-\omega_T(t-t')}{2Q}} \tag{22}$$
$$\times\, v_{s'}(t')\cos\frac{\pi k z_s}{\ell}\cos\frac{\pi k z_{s'}}{\ell}\, J'_\mu\!\left(\frac{\pi\beta r_s}{a}\right) J_\mu\!\left(\frac{\pi\beta r_{s'}}{a}\right).$$

This equation gives first-order blowup only for $\mu = 1$. Now we integrate Eq. (22) over the time the sth bunch is in a particular cavity. Defining $t_s$ to be the time at which the sth bunch passes the center of the cavity, we introduce the time variable $\tau$;

$$t = t_s + \tau$$

$$z_s = v_s\tau + \ell/2.$$

The integral over the times that the sth and s′th bunches are in the cavity is then, for $Q \gg 1$,

$$\int d\tau \int d\tau' \,\sin\omega_T(\tau-\tau')\cos\frac{\pi k z_s}{\ell}\cos\frac{\pi k z_{s'}}{\ell}$$
$$= \sin\omega_T(t_s - t_{s'})\int d\tau\int d\tau' \cos\omega_T(\tau-\tau')\cos\frac{\pi k z_s}{\ell}\cos\frac{\pi k z_{s'}}{\ell}$$
$$= \sin\omega_T(t_s-t_{s'})\,\frac{\ell}{v_s}\,\frac{\ell}{v_{s'}}\,T_s\,T_{s'}\,, \tag{23}$$

which introduces the usual transit-time factors $0 < T_s < 1$. Henceforth we set $v_s = v_{s'} = v$; $T_s = T_{s'} = T$. Equation (22) then gives, for the transverse impulse imparted to bunch s during its traversal of the cavity,

$$\Delta p_s = \left(\frac{e^2}{4\pi\epsilon_o a^2\ell}\right)\frac{4}{\omega_T}\left(\frac{\omega_c}{\omega_T}\right)^4\left(\frac{\ell}{\lambda}\right)^2\frac{T^2}{J_2^2(\pi\beta)}\;\frac{1}{1+\delta_{k,o}}$$
$$\times\sum_{s'}\sin\omega_T(t_s-t_{s'})\; e^{\frac{-\omega_T(t_s-t_{s'})}{2Q}}\; r_{s'}(t_{s'}), \tag{24}$$

where $\omega_c = \pi\beta c/a$ and $\lambda = c/\omega_T$.

D. Image Charges in the Wall and Energy Transfer

Figure 3 shows the electric field lines inside a long conducting tube with a line charge parallel to the axis. The lines are circles, terminating in an image line charge outside. The radial electrostatic force on each length $\ell$ of line charge, containing e units of charge, and off-axis r units, is

$$\frac{e^2 r}{4\pi\epsilon_o a^2\ell} = gr, \tag{25}$$

which defines g, the quantity in the first parentheses of Eq. (24).

Some more of the factors in Eq. (24) can be understood in either of two ways. First, consider a stationary charge e in a cylindrical cavity, at position $\vec{r}$. The potential at $\vec{r}\,'$ is[3]

$$\varphi(\vec{r}\,') = \frac{4e}{\pi\epsilon_o\ell a^2}\sum_{\mu k t}\frac{\cos\mu(\varphi'-\varphi)}{J^2_{\mu+1}(\pi\beta_{\mu t})(1+\delta_{\mu,o})(1+\delta_{k,o})}$$
$$\times\frac{\sin\frac{\pi k z'}{\ell}\sin\frac{\pi k z}{\ell}\,J_\mu\!\left(\frac{\pi\beta_{\mu t}r'}{a}\right)J_\mu\!\left(\frac{\pi\beta_{\mu t}r}{a}\right)}{\left(\frac{\pi k}{\ell}\right)^2+\left(\frac{\pi\beta_{\mu t}}{a}\right)^2}. \tag{26}$$

The radial force is then, keeping only the term in the sum which has the same symmetry as our deflecting mode,

$$-\,e\,\partial\varphi/\partial r' = 4g\left(\frac{\omega_c}{\omega_T}\right)^2\frac{r}{(1+\delta_{k,o})\,J_2^2(\pi\beta)}\,. \tag{27}$$

These electrostatic analogies should not be taken too seriously, of course, because in the real system magnetic forces are usually dominant.

Alternatively, the factors can be expressed in terms of stored energy and energy transfer. The time average stored energy for a vector potential given by Eq. (12) is

$$W_s = \frac{1}{\mu_o}\left(\frac{\pi\beta}{a}\right)^2. \tag{28}$$

The energy transfer from the deflecting mode to a bunch is

$$W_B = r_s W_B' = \int e\,\mathcal{E}_z v\,dt, \tag{29}$$

the maximum value of which is

$$W_B' = \frac{e\ell c^2 T}{2\omega_T} \left(\frac{\pi\beta}{a}\right)^3 A_{\mu kt}. \qquad (30)$$

Therefore,

$$\frac{(W_B')^2}{W_s} = 4g\left(\frac{\omega_c}{\omega_T}\right)^4 \left(\frac{\ell}{\lambda}\right)^2 \frac{T^2}{J_2^2(\pi\beta)} \frac{1}{1+\delta_{k,o}} \qquad (31)$$

which is nearly identical with the coefficient in Eq. (24).

The usual definition of the deflecting mode shunt impedance $r_t$,

$$\frac{r_t T^2}{Q} = \frac{\left(\lambda W_B'/e\right)^2}{L\,\omega_T W_s}, \qquad (32)$$

enables one to write the deflection equation with $r_t/Q$ as the principal part of the coefficient, also.

E. Discussion

The deflections described by Eq. (24) may be looked upon roughly as follows. Consider that as each bunch passes through the cavity it induces an image charge in the wall, which subsequently oscillates and decays. The sth bunch then sees a coherent sum of the images of the preceding bunches; their number limited either by Q or the pulse length, their phases controlled by the elapsed time and the displacement of the preceding bunch. Resonance between the bunch frequency and $\omega_T$ is not necessary for coherence; in fact, if the deflecting mode frequency is any integral or half-integral multiple of the bunch frequency, Eq. (24) vanishes identically! This rather surprising fact may be understood in the following way.

F. The $TM_{110}$ Mode

Although Eq. (24) is valid for all the deflecting modes (i.e. the TM modes with $\mu = 1$ shown in Eq. (16); the TE modes do not deflect[4]) we will now take the $TM_{110}$ as an easily visualized example. It has no transverse components of E anywhere, and a maximum of B on the axis. The deflection is therefore entirely due to the magnetic force; the energy transfer from the beam is, as always, due to the work done by the off-axis beam on the longitudinal E-field.

A bunch s′ which passes the center of the cavity at a time $t_s'$ at which E is at its peak transfers a maximum amount of energy to the deflecting mode. This implies that the increment $\delta E_s'$ contributed by the bunch is in phase with E.

Now, suppose a subsequent bunch s arrives an integral number of half-periods later; i.e.,

$$\omega_T\,(t_s - t_s') = n\pi,\ n = 1, 2, 3, \ldots \quad .$$

Then, depending on whether n is even or odd, the contribution of s to E, $\delta E_s$, will be in phase or 180° out of phase with $\delta E_s'$. But, because the contribution $\delta B_s'$ to B of bunch s′ is 90° out of phase with $\delta E_s'$, bunch s will be undeflected by the contributions of bunch s′ to the cavity fields.

The next step in the argument is to recognize that $\delta E_s'$, being a field produced by a prescribed current, is independent of the field which was already in the cavity when bunch s′ passed through. Thus, the presence of the $\sin \omega_T(t_s - t_s')$ factor in Eq. (24) is plausible, and exact resonance between the deflecting mode frequency and a bunch frequency harmonic is, in fact, incompatible with beam blowup.

## IV. Deflecting Modes in Neighboring Cells Coupled Together

The model we have just considered can be generalized to include cell-to-cell coupling. Using an equivalent circuit approach, we will now derive the deflection equation for the coupled chain, and identify the coefficients by taking the limit of zero coupling and comparing with the results of the preceding section.

Rather than considering the cavity fields in detail, we now assume that only one deflecting mode is important, and take its amplitude in each cell to be the dynamical variables which describe the fields. These variables can be identified with charges or currents in the equivalent circuit shown in Fig. (4). The coördinates describing the beam are still, of course, the bunch positions and velocities. The beam-equivalent circuit interaction must be chosen in such a way as to reproduce the form of the beam deflection equations obtained in Sect. III in the limit of no cell-to-cell coupling.

A. Choice of Lagrangian

A dependable way to get a consistent set of equations is to derive them from a Lagrangian. This we divide into 3 parts. First, for the chain of circuits,

$$\mathcal{L}_C = \sum_n \left(\frac{1}{2} L_o \dot{q}_n^2 - \frac{1}{2C} q_n^2\right) + \frac{1}{2} \sum_{n,m} L_{nm} \dot{q}_n \dot{q}_m \qquad (33)$$

the choice is straightforward, as it is for the beam:

$$\mathcal{L}_B = \sum_s \frac{1}{2} M \left(\dot{z}_s^2 + \dot{x}_s^2\right). \qquad (34)$$

From Sect. III we know that the beam-cavity interaction depends linearly on the field, the bunch velocity, and the bunch deflection. The simplest assumption for the interaction Lagrangian, then, is

$$\mathcal{L}_{CB} = \sum_m g_m\, d_m\, \dot{q}_m, \tag{35}$$

where

$$d_m = \sum_s l_m(z_s)\, \dot{z}_s\, x_s \tag{36}$$

is the transverse moment of the beam current in the mth cell at time t, and $g_m$ is a coupling constant proportional to the charge. Here $l_m(z_s)$ is unity when the sth bunch is inside the mth cell, zero otherwise. The only choice we had to make in writing Eq. (35) was whether to use q or $\dot{q}$. Comparison of subsequent equations with those of Sect. III determined that choice.

B. Equations of Motion

The whole Lagrangian is the sum of Eqs. (33), (34), and (35); the Eulerian equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} = 0 \tag{37}$$

yield the equations of motion for x and q:

$$L_o \ddot{q}_n + q_n/C + \sum_m L_{nm} \ddot{q}_m + \frac{d}{dt}(d_n g_n) = 0, \tag{38}$$

$$M\, \ddot{x}_s - \sum_m g_m\, \dot{q}_m\, l_m(z_s)\, \dot{z}_s = 0. \tag{39}$$

As before, our procedure is to eliminate the fields from these equations by solving Eq. (38) for $\dot{q}_n$ and substituting the result into Eq. (39), yielding the generalization of the deflection Eq. (24).

A Green's function for the chain is defined as the solution of

$$\left(\frac{\omega_o^2}{\omega^2} - 1\right) G_{nn'}(\omega) - \sum_m k_{nm}\, G_{mn'}(\omega) = \delta_{nn'} \tag{40}$$

which satisfies boundary conditions appropriate to the nature of the end cells. Here we have put $\omega_o^{-2} = L_o C$, $k_{nm} = L_{nm}/L_o$. For the case where $L_{nm}$ only couples adjoining cells (i.e. $k_{nm} = k\delta_{n,m\pm1}$) Eq. (40) is easy to solve for G in closed form. For our present purposes, though, the usual normal mode expansion is more convenient. We therefore define a set of normal modes by

$$\left(\frac{\omega_o^2}{\omega_q^2} - 1\right) X_n^{(q)} - \sum_m k_{nm}\, X_m^{(q)} = 0 \tag{41}$$

Completeness and orthonormality require that

$$\sum_q X_n^{(q)}\, X_{n'}^{(q)*} = \delta_{nn'},$$
$$\sum_n X_n^{(q)*}\, X_n^{(q')} = \delta_{qq'}. \tag{42}$$

The solution of Eq. (40) is then

$$G_{nn'}(\omega) = \omega_o^{-2} \sum_q \frac{X_n^{(q)}\, X_{n'}^{(q)*}}{\omega^{-2} - \omega_q^{-2}}, \tag{43}$$

which can be used to construct a solution of Eq. (38). After a change of variable to

$$\dot{\xi}_n(t) = q_n(t) = i\int \omega\, \xi_n(\omega) e^{i\omega t}\, d\omega, \tag{44}$$

it is

$$\xi_n(\omega) = -\frac{1}{L_o \omega^2} \sum_m G_{nm}(\omega)\, g_m d_m(\omega), \tag{45}$$

where

$$d_m(\omega) = \frac{1}{2\pi}\int dt\, e^{-i\omega t}\, d_m(t). \tag{46}$$

C. Deflection Equation

The longitudinal motion of the beam bunches is considered to be prescribed. That is, the functions

$$v_{sm}(t) = l_m(z_s)\, \dot{z}_s(t)$$
$$= \int v_{sm}(\omega)\, e^{i\omega t}\, d\omega \tag{47}$$

are known. For the simplest case in which the bunches are points of charge with constant longitudinal velocity v and constant spacing $vt_B$,

$$v_{sm}(\omega) = \frac{\ell T_B}{2\pi} e^{-i\omega(st_B - mL/v)}, \tag{48}$$

in which

$$T_B = \frac{\sin \ell\omega/2v}{\ell\omega/2v} \qquad (49)$$

will be part of the transit time factor.

Now Eq. (45) becomes

$$\xi_n(\omega) = -\frac{1}{L_o\omega^2} \sum_{s,m} G_{nm}(\omega) g_m \int v_{sm}(\omega-\omega') x_s(\omega') d\omega', \qquad (50)$$

and Eq. (39) can be written in terms of its Fourier components as follows:

$$L_o M\omega^2 x_s(\omega) = -\sum_m g_m \int d\omega' v_{sm}(\omega-\omega') \sum_{m'} G_{mm'}(\omega')$$

$$\times\, g_{m'} \sum_{s'} \int v_{s'm'}(\omega'-\omega'')\, x_{s'}(\omega'')\, d\omega'', \qquad (51)$$

where we have substituted $\dot{q} = \ddot{\xi}$ from Eq. (50). This is the deflection equation, to be solved for $x_s$.

D. Graphical Interpretation

As before, a proper choice of pole displacement and contour in the $\omega'$ integration will insure that causality is satisfied and that energy is dissipated in the cavities at the proper rate. Figure 5 shows space-time tracks of a sequence of bunches, and the interaction of one bunch with another via the deflecting mode, according to Eq. (51). A bunch $s'$ in cell $m'$, oscillating transversely with frequency $\omega''$ combines with the component of the longitudinal time structure of the beam with frequency $\omega'-\omega''$ to excite a deflecting mode with frequency $\omega'$. This deflecting mode then propogates from cell $m'$ to cell $m$ (via $G_{mm'}$), combines again with the longitudinal structure of the beam ($v_{sm}$) to excite another frequency component of the displacement of the sth bunch in cell m. The process can then be repeated, corresponding to an iteration of Eq. (51), which is one way to proceed to obtain the solution of the integral equation.

E. Further Reduction

Now we denote by

$$t_{sm} = st_B + mL/v \qquad (52)$$

the time at which the sth bunch passes the center of the mth cell. Then, if we take the inverse transform of Eq. (51), use Eq. (48), and integrate both sides over the time interval the sth bunch is in the mth cell, we get

$$\Delta p_s(m) = \frac{(\ell T_B g)^2}{2\pi L_o} \sum_{m',s'} \int d\omega'\, G_{mm'}(\omega') \qquad (53)$$

$$\times\; e^{i\omega'(t_{sm}-t_{s'm'})}\; x_{s'}(t_{s'm'})$$

for the momentum transferred in the mth cell to the sth bunch ($\Delta p_s(m)$).

There are two cases in which it is easy to proceed further in the analysis. First, if all the frequencies of the chain of cavities are equal ($\omega_q = \omega_o$), corresponding to zero bandwidth or no cell-to-cell coupling, then, by Eq. (43)

$$G_{mm'}(\omega) = \omega_o^{-2} \frac{\delta_{mm'}}{\omega^{-2} - \omega_o^{-2}} \qquad (54)$$

and Eq. (53) should become equivalent to Eq. (24). We can then establish a correspondence between our circuit parameters and those of the cylindrical cavity.

Upon inserting Eq. (54) into Eq. (53), and performing the $\omega'$ integral with poles displaced into the upper half-plane as before, Eq. (53) becomes

$$\Delta p_s(m) = \frac{(\ell T_B g)^2}{L_o} \omega_o \sum_{s'<s} \sin \omega_o(t_s-t_{s'}) e^{\frac{-\omega_o(t_s-t_{s'})}{2Q}}$$

$$\times \quad x_{s'}(t_{s'}) \; . \qquad (55)$$

This equation is identical with Eq. (24) if g is chosen to be that function of the cavity geometry which makes

$$\frac{(\omega_o \ell T_B g)^2}{L_o} = \frac{(W_B')^2}{W_s} \; . \qquad (56)$$

The other tractable situation occurs when the modes $\omega_q$ are spaced sufficiently far apart ($\Delta\omega_q > \omega_q/2Q$) that only one at a time contributes significantly to G. Then

$$G_{mm'} = \omega_o^{-2} \frac{\chi_m^{(q)}\, \chi_{m'}^{(q)*}}{\omega^{-2} - \omega_q^{-2}} \; , \qquad (57)$$

and the $\chi$'s, for convenience taken to satisfy periodic boundary conditions in a chain of length N, are

$$\chi_m^{(q)} = N^{-1/2} e^{iqm} \; , \qquad (58)$$

with $q = 2\pi/N,\ 2 \cdot 2\pi/N,\ 3 \cdot 2\pi/N \cdots 2\pi$. Now Eq. (53) becomes

$$\Delta p_s(m) = \frac{\omega_q^3 (W_o')^2}{\omega_o^4 N W_s} \sum_{m',s'} \sin \omega_q(t_{sm} - t_{s'm'}) \qquad (59)$$

$$\times\ e^{-\frac{\omega_q(t_{sm} - t_{s'm'})}{2Q}}\ e^{iq(m-m')} x_{s'}(t_{s'm'}),$$

where the sum is restricted so that $t_{sm} > t_{s'm'}$.

## F. Phase Coherence

Equation (59) contains two sums, one over cell number, one over bunches. Crucial to the phenomenon of beam-blowup are the questions of whether the terms in these sums add coherently or cancel one another, and whether coherence in one section precludes coherence in the next, where the structure is slightly different. To facilitate discussion of this point we expand $x_{s'}$ as follows

$$x_{s'}(t_{s'm'}) = \int e^{is'\sigma} e^{im'\mu} x(\sigma,\mu)\, d\sigma d\mu. \qquad (60)$$

We can now carry out the $m'$, $s'$ sums; the result, however, is complicated. Its significance lies in the fact that they are small unless

$$\sigma \pm \omega_q t_B = 0,\ \pm 2\pi,\ \pm 4\pi, \ldots \qquad (61)$$

to within about $\pi$/(Number of bunches in pulse) or $\pi/Q$, whichever is bigger; and

$$\mu - q \pm \omega_q L/v = 0,\ \pm 2\pi,\ \pm 4\pi, \ldots \qquad (62)$$

to within $\pi$/(Number of cells in a section). A component $x(\sigma,\mu)$ for which these criteria are satisfied will be amplified on passing through the section, and the pulse structure will acquire wiggles. Because the blowup length is always large compared to a section length, $\mu$ is always small compared to q. A snapshot of the pulse will show modulation in the bunch displacements built up of wave numbers $\sigma$ satisfying Eq. (61). If the same $\sigma$ is not favored by successive sections, $x(\sigma, 0)$ will die out and no blowup will occur. Figure 6 is an attempt at elucidation of the situation. The fuzziness of the q-criterion Eq. (62) is translated into a spread in the phase velocity (slope of diagonal line) of a few percent. The $\omega_q$-criterion Eq. (61) is much more stringent - a few parts in $10^4$, say. $x(\sigma,0)$ will blow up if a long enough string of successive sections have modes in their deflecting bands which lie on the $\omega = \omega_q$ line with phase velocities within the fan centered on the bunch velocity in Fig. 6. Whether or not this will happen depends on how the modifications in the structure from section to section shift the deflecting mode dispersion curve.

## G. Discussion

Small bandwidth is obviously good; there can be no blowup in a chain of uncoupled cells unless the variation in structure is very small indeed. For fixed bandwidth a change in shape or shift in frequency would be beneficial if long series of consecutive sections were thereby prevented from coöperating in amplifying a component $x(\sigma,0)$ of the deflections. Blowup can also be suppressed by adjusting the deflecting mode frequency to be at or near an integral multiple of the bunch frequency, as discussed in Sect. III F. It must be very near indeed, however, for suppression to occur, because the sum over $s'$ in Eq. (59), if Eq. (61) is satisfied, is proportional to the number (R) of terms it contains unless $\omega_q t_B$ is within $R^{-1}$ of a multiple of $\pi$.

Our blowup Eq. (59) as yet contains no focussing force, which has an important effect and should be included. Equation (59) (or Eq. (24) in the uncoupled case) does not lend itself readily to solution by the usual means[1,5] because the presence of the sine factor in the sum makes it hard to justify the replacement of the latter by an integral.

† Work performed under the auspices of the U. S. Atomic Energy Commission.

## References

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DISCUSSION

W. M. VISSCHER, LASL

MILLS, MURA: None of the theoretical discussions of beam breakup have included the electron particle dynamics to the extent that they treat the effects of transverse velocity spread, or what is usually called Landau damping. In circular machines, Landau damping has been effective in suppressing these instabilities. On the other hand, circular machines already have strong mechanisms which produce velocity spread. It may be that a relatively small correction to the transverse focusing fields, say by octupoles, may produce striking effects in the linac beams, which are practically devoid of transverse velocity spread.

PANOFSKY, SLAC: I believe that Landau damping here would be analogous to calculating the effect of introducing octupole lenses, which I believe Loew referred to. This would destroy the phase coherence between particles starting at different radii. A computer program for this calculation is being written by Herrmannsfeldt. However, initial indications seem to show that these octupoles do not do as much good as the back-of-an-envelope calculation would indicate. Another analogous effect which we investigated, is to use time-varying quadrupoles. For example, one may take a quadrupole, which is opposed to a regular quadrupole, but reverse its sign every tenth of a μsec. Hence, the magnetic field would be reversed ten times during one μsec pulse. Dr. Helm programmed this effect into his calculation and obtained some improvements. However, they do not seem large enough to make this solution look very interesting from a practical point of view. We will look further into the octupole problem.

HELM, SLAC: I think the whole problem on both these nonlinear and modulated quadrupole mechanisms is that you can't make a very big perturbation in any one focusing period. Another point on the analogy of Landau damping is that it requires finite transverse phase space in order to give incoherent effect on the blowup interaction. We have finite transverse phase space, and when you put in a strong enough nonlinear element to do any good, it blows the beam up by nonlinear defocusing effects. In the circular machine, I think these effects work very well because the instability is taking place over many, many betatron wave lengths, while in our case we have an e-folding within a sector or so of the machine.

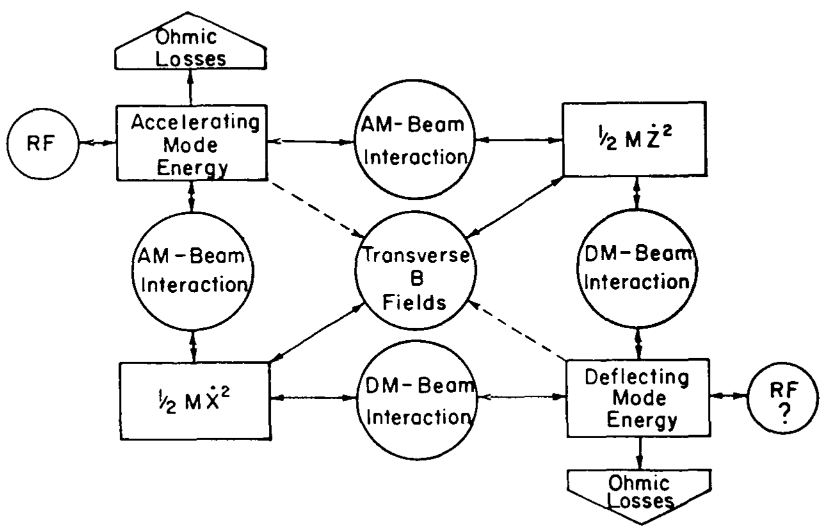


Fig. 1. Energy flow among the various degrees of freedom in the linac-beam system is shown by the solid lines.

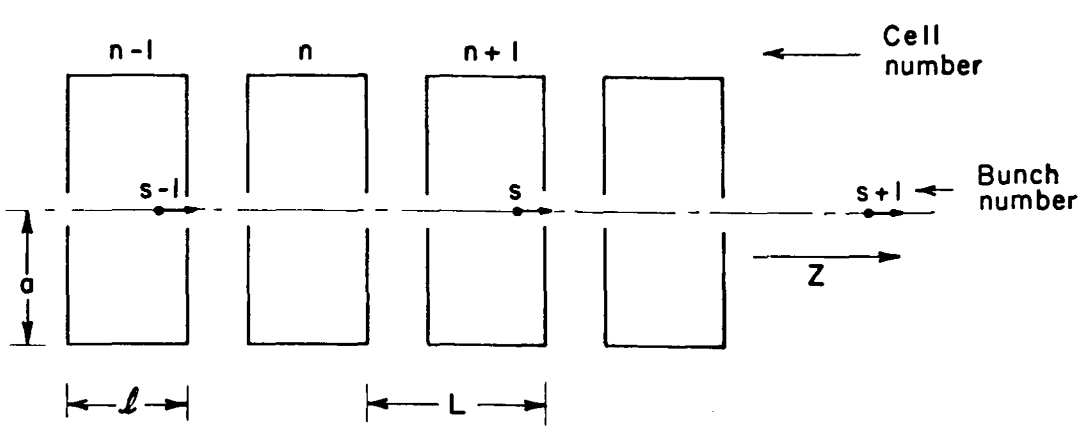


Fig. 2. Uncoupled chain of cylinders with small holes as a linac model.

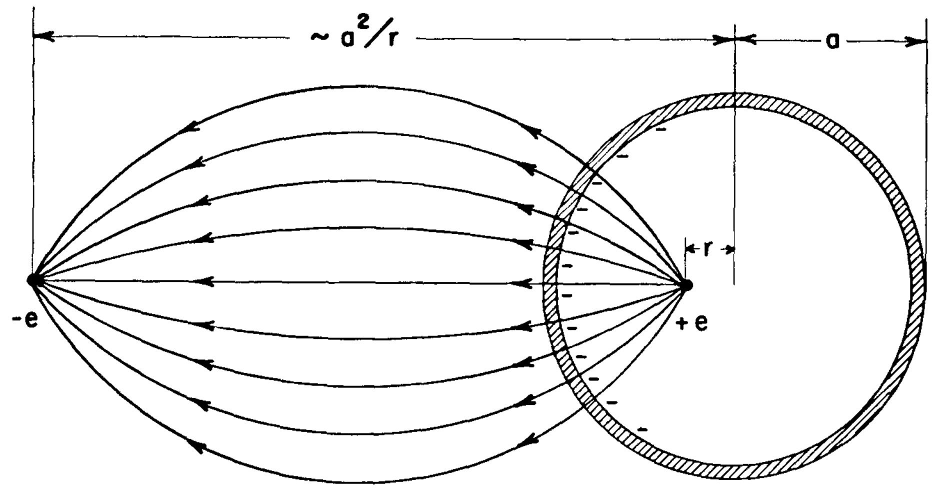


Fig. 3. Off-axis line-charge in a long cylinder is attracted by its image in the wall.

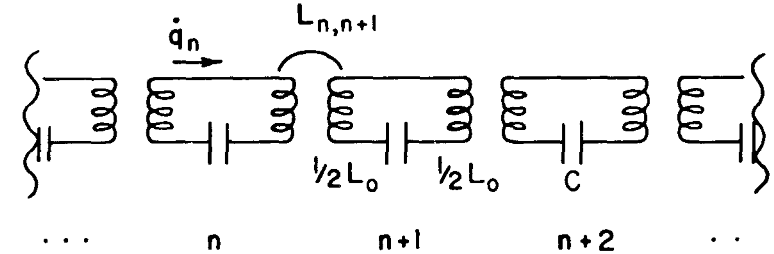


Fig. 4. Equivalent circuit for deflecting modes in a linac model with cell-to-cell coupling.

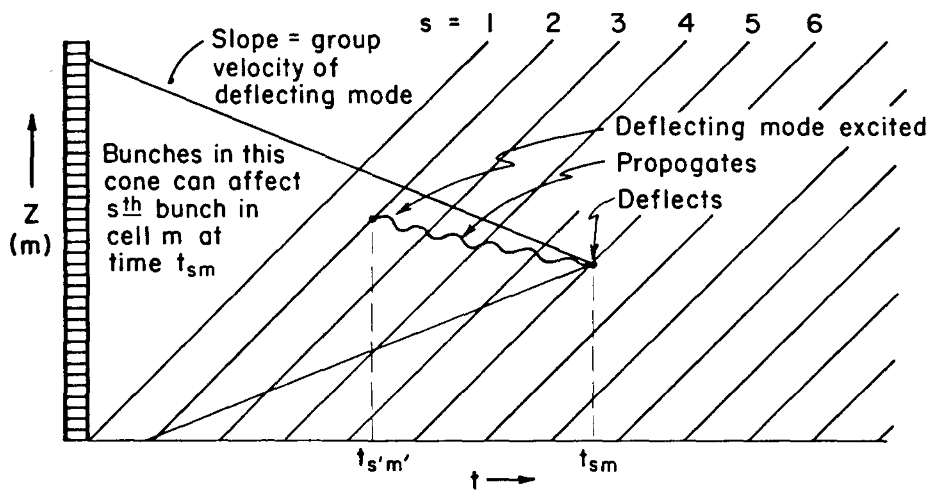


Fig. 5. World lines of bunches in linac interacting causally.

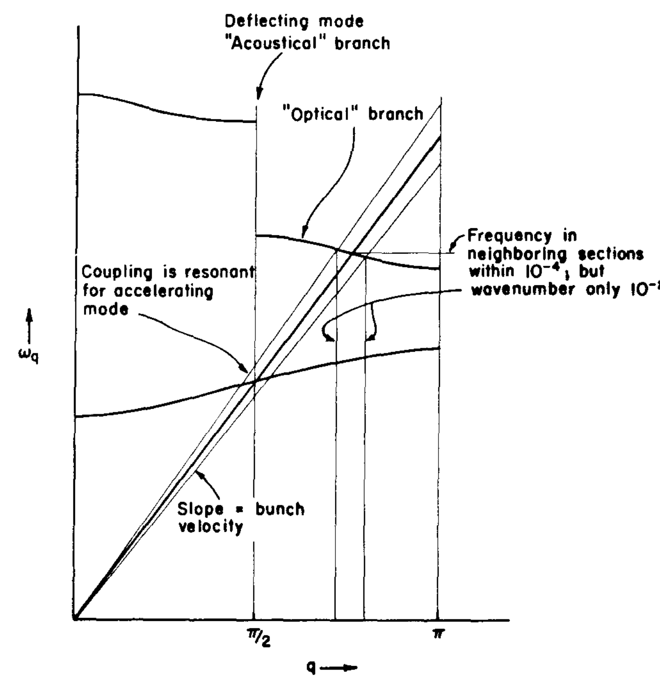


Fig. 6. Brillouin diagram for accelerating, deflecting modes in a $\pi/2$ structure.