THE EFFECT OF QUADRUPOLE FRINGING FIELDS ON COUPLING IN LINACS*

Robert L. Gluckstern
Physics Department, University of Massachusetts
Amherst, Massachusetts

I. Introduction

The effect of quadrupole fringing fields on transverse emittance has recently been pointed out by Regenstreif.[1] He finds serious aberrations which must be taken into account in the design of transport systems. The purpose of this note is to examine the effects of quadrupole fringing fields in linacs on the transverse motion.

II. Quadrupole Fringing Fields

It is convenient to use a scalar potential ψ to describe the quadrupole focusing field. For a potential

$$\psi = \alpha xy \tag{1}$$

one obtains

$$H_x = \alpha y \ , \quad H_y = \alpha x \tag{2}$$

which is the usual representation of the fields in which the x and y motions are separable.

If α depends (as it does) on the axial coordinate, s, additional field components are obtained. Moreover, the scalar potential αxy is no longer a solution of the Laplace equation. In fact one writes[1] a series in powers of x and y:

$$\psi = \alpha(s)xy - \alpha''(s)(x^3y + xy^3)/12 + \ldots \tag{3}$$

and obtains

$$\begin{aligned} H_x &= \alpha(s)y - \alpha''(s)(3x^2y + y^3)/12 \\ H_y &= \alpha(s)x - \alpha''(s)(x^3 + 3xy^2)/12 \\ H_z &= \alpha'(s)xy - \alpha'''(s)(x^3y + xy^3)/12 \ . \end{aligned} \tag{4}$$

The main effect of the additional terms in Eq. (4) on the x or y motion comes from the first term in H_z and the second term in H_x and H_y. Specifically, one obtains an additional force

$$\Delta F_x = \frac{ev}{c}[\alpha' xy\ y' + \alpha''(\frac{3xy^2 + x^3}{12})] \tag{5}$$

$$\Delta F_y = -\frac{ev}{c}[\alpha' xy\ x' + \alpha''(\frac{3x^2y + y^3}{12})]. \tag{6}$$

The corresponding impulse in the x direction at the entrance to a magnet can then be obtained from Eq. (5) by writing

$$\Delta x' = \frac{1}{mc\beta\gamma}\int dt\ \Delta F_x \simeq$$

$$\simeq \frac{e}{mc^2\beta\gamma}\int ds[\alpha' xy\ y' + \alpha''(\frac{3xy^2 + x^3}{12})]. \tag{7}$$

We shall assume that the fringing field is confined to a narrow axial region. As a consequence x, x', y, y' will not vary appreciably in the region of the fringing field and α' and α'' can be taken to be proportional to the Dirac delta function and its derivative. Writing

$$\alpha'(s) = \alpha\delta(s - s_e), \quad \alpha''(s) = \alpha\delta'(s - s_e) \tag{8}$$

where α is the (constant) central value of the magnetic field gradient, one has, from the properties of $\delta(s - s_e)$ and $\delta'(s - s_e)$,

$$\begin{aligned} \int_{s_e-\varepsilon}^{s_e+\varepsilon} ds G(s)\alpha'(s) &= \alpha G(s_e), \\ \int_{s_e-\varepsilon}^{s_e+\varepsilon} ds G(s)\alpha''(s) &= -\alpha G'(s_e)\ , \end{aligned} \tag{9}$$

from which

$$\Delta x' \simeq \frac{e\alpha}{mc^2\beta\gamma}\left[\frac{xy\ y'}{2} - \frac{x'y^2}{4} - \frac{x'x^2}{4}\right]_{s_e} . \tag{10}$$

Setting

$$K = \frac{e\alpha}{mc^2\beta\gamma} \tag{11}$$

one has

$$\Delta x' = \pm K[\frac{xy\ y'}{2} - \frac{x'y^2}{4} - \frac{x'x^2}{4}] \tag{12}$$

$$\Delta y' = \mp K[\frac{xy\ x'}{2} - \frac{y'x^2}{4} - \frac{y'y^2}{4}] \tag{13}$$

at the entrance to and exit from the quadrupole, while the equations of motion within the quadrupole are

$$(\beta\gamma)^{-1} \frac{d}{ds}(\beta\gamma \frac{dx}{ds}) + Kx = 0 \quad (14)$$

$$(\beta\gamma)^{-1} \frac{d}{ds}(\beta\gamma \frac{dy}{ds}) - Ky = 0 \quad (15)$$

III. Amplitude Variation and Emittance Change

The impulses in Equations (12) and (13) cause the emittance invariant in each direction to change. If one writes, in a notation similar to that of Courant and Snyder,[2]

$$W_x = \beta\gamma(\gamma_x x^2 + 2\alpha_x xx' + \beta_x x'^2) \quad (16)$$

one has, at the entrance to, and exit from each magnet

$$\Delta W_x \simeq 2\beta\gamma\Delta x'(\alpha_x x + \beta_x x'). \quad (17)$$

It is possible to evaluate the sum of the two impulses in a single magnet exactly by expressing all the parameters at the end of each magnet in terms of the corresponding values at the center of the magnet. We have done this for the xy y' term in $\Delta x'$ in a NSNS configuration and find that the approximate result for the smoothed approximation is sufficiently accurate for our needs. For this reason we shall use only the smoothed approximation in which one writes, for the focusing magnet in a NSNS configuration,

$$\Delta W_x \simeq 2\beta\gamma\, \beta_o x_o'(\Delta x'_{entrance} + \Delta x'_{exit}), \quad (18)$$

where the subscript 0 refers to the center of the magnet. Using Eq. (12) one has

$$\Delta W_x \simeq - 2\beta\gamma\, \beta_o x_o' K\left[\frac{xy\, y'}{2} - \frac{x'(x^2+y^2)}{4}\right]_{entrance}^{exit} \quad (19)$$

We shall further approximate the term in the bracket by making use of the fact (consistent with the smoothed approximation) that the main variation from entrance to exit is in the terms x' and y'. Writing

$$x'_{exit} - x'_{entrance} \simeq x_o'' \ell = - Kx_o \ell$$
$$y'_{exit} - y'_{entrance} \simeq y_o'' \ell = Ky_o \ell \ , \quad (20)$$

where ℓ is the magnet length, one finally obtains

$$\Delta W_x^o \simeq - \frac{1}{2} \beta\gamma\beta_o x_o' K^2 \ell [3x_o y_o^2 + x_o^3] \quad (21)$$

$$\Delta W_y^o \simeq - \frac{1}{2} \beta\gamma\beta_1 y_o' K^2 \ell [3y_o x_o^2 + y_o^3] \quad (22)$$

for the emittance changes at a focusing (in the x direction) magnet. It is important to note that the signs in Equations (21) and (22) are the same even though Equations (12) and (13) are opposite in sign. This can be traced to Eq. (20).

In a similar way one can write for the emittance change at a defocusing magnet

$$\Delta W_x^1 \simeq - \frac{1}{2} \beta\gamma\beta_1 x_1' K^2 \ell [3x_1 y_1^2 + x_1^3] \quad (23)$$

$$\Delta W_y^1 \simeq - \frac{1}{2} \beta\gamma\beta_o y_1' K^2 \ell [3y_1 x_1^2 + y_1^3] \quad (24)$$

where the subscript 1 refers to the center of the defocusing magnet. One can further express x_o, x_o', y_o, y_o', x_1, x_1', y_1, y_1' in terms of the amplitude and phase of the transverse motion. Specifically, one writes

$$x_o = A_x \sin\phi_x^o, \quad x_1 = A_x \sin\phi_x^1 \sqrt{\frac{\beta_1}{\beta_o}}$$
$$x_o' = \frac{A_x \cos\phi_x^o}{\beta_o}, \quad x_1' = \frac{A_x \cos\phi_x^1}{\sqrt{\beta_o\beta_1}} \quad (25)$$

$$y_o = A_y \sin\phi_y^o \sqrt{\frac{\beta_1}{\beta_o}}, \quad y_1 = A_y \sin\phi_y^1$$
$$y_o' = \frac{A_y \cos\phi_y^o}{\sqrt{\beta_o\beta_1}}, \quad y_1' = \frac{A_y \cos\phi_y^1}{\beta_o} \quad (26)$$

Equation (21) then becomes

$$\Delta W_x^o = \frac{\beta\gamma K^2 \ell A_x^2}{16} \{3 \frac{\beta_1}{\beta_o} A_y^2[\sin(2\phi_x^o - 2\phi_y^o) + \sin(2\phi_x^o + 2\phi_y^o)] + A_x^2 \sin 4\phi_x^o + 2(A_x^2 + 3\frac{\beta_1}{\beta_o} A_y^2) \sin 2\phi_x^o\} \quad (27)$$

with similar expressions for ΔW_y^o, ΔW_x^1, ΔW_y^1.

Our task now is to sum Eq. (27) and the corresponding one for ΔW_x^1 applied at each magnet of the linac. The accumulation of these terms will depend on the phase advance of $2\phi_x$, $4\phi_x$, $2\phi_x \pm 2\phi_y$ from one magnet to the next. Since the coefficients will be slowly varying from one magnet period to the next, contributions can be expected to cancel unless there is some sort of resonance. But there is indeed, since in all linac designs μ_x is taken to be the same as μ_y. One therefore expects accumulation of the terms where the phase $2\phi_x - 2\phi_y$ remains constant, at its initial value $2a_x - 2a_y$. One therefore has approximately

$$\Delta W_x \simeq A_x^2 A_y^2 \sin (2a_x - 2a_y) \sum_i (\frac{3\beta_1}{8\beta_o}) \beta\gamma K^2 \ell \quad (28)$$

where the contribution from ΔW_x^1 is found to be equal to that of ΔW_x^o. The sum is now over magnet periods.

In a similar way one finds

$$\Delta W_y \simeq A_x^2 A_y^2 \sin (2a_y - 2a_x) \sum_i (\frac{3\beta_1}{8\beta_o}) \beta\gamma K^2 \ell . \quad (29)$$

Even though we have here used the smoothed approximation, the sums in Eqs. (28) and (29) are identical for the exact strong focused oscillation.

One can put Equations (28) and (29) into a more convenient form by writing $\theta^2 = K\ell^2$, where θ is the "angle" of the transverse motion in a magnet. The sums can also be approximated by assuming a design for which θ^2 and the ratio of β_1/β_o remain constant as β increases. The summand then decreases as β^{-2} and the sum to infinity converges. One then obtains

$$\Delta W_x = - \Delta W_y \simeq A_x^2 A_y^2 \sin (2a_x - 2a_y) (\frac{3\beta_1}{8\beta_o}) \beta\gamma (\frac{\theta^4}{\ell^3})(\frac{\beta_{inj}}{2\Delta\beta}) . \quad (30)$$

The last factor contains the velocity at injection as well as the (constant) velocity change per cell, and represents the evaluation of the sum. All other factors are to be taken at injection.

IV. Transverse Beam Changes

If one starts with a collection of particles each having the same value of W_x and W_y, but differing values of a_x and a_y, there will be particles for which each emittance, and therefore the maximum amplitude of transverse oscillation in each direction, will increase. The order of magnitude of the amplitude increase is 20%, as confirmed by computations.[3] It is obvious, however, that this situation corresponds to a beam of rectangular xy cross section, which is unrealistic. A beam of axial symmetry has a distribution in the four-dimensional phase space which is a function of $W_x + W_y$ and has approximately circular xy cross section. For such a beam there is no change of the radial coordinate, since ΔW_x and ΔW_y cancel. In fact the distortions represented by Eq. (30) are merely rotations about the axial direction. This phenomenon is confirmed in the numerical computations.[3]

For completeness, we will obtain the maximum distortion represented by Eq. (30). Since

$$A_x^2 = \frac{\beta_o}{\beta\gamma} W_x \quad (31)$$

one has

$$\left.\frac{\Delta A_x}{A_x}\right|_{max} \simeq \frac{3}{16} \beta_1 (\frac{\theta^4}{\ell^3}) A_y^2 (\frac{\beta_{inj}}{2\Delta\beta}) . \quad (32)$$

The results obtained here are approximate, in the sense that we have neglected contributions from the $2\phi_x$, $4\phi_x$, $2\phi_x + 2\phi_y$ terms in Eq. (27). Since these phases vary rapidly, the contribution of these terms will only be of the order of magnitude of the first term. In the unlikely event that the design is one for which $\mu_x = \mu_y = 90°$, however, the phase advance per magnet period of $4\phi_x$ or $2\phi_x + 2\phi_y$ will be 360° and these terms will accumulate, as they did for the $2\phi_x - 2\phi_y$ terms. This accumulation will not be as large though, since the impulses in the focusing and defocusing magnets will differ in phase by 180°. Although this cancellation is complete only in the smoothed approximation, the "resonance" can be considered unimportant.

V. Summary and Conclusions

We have obtained the distortion of the transverse phase spaces due to fringing fields in the magnets of the strong focusing system. For a beam with fixed circular symmetry there is essentially no change in the radial oscillation amplitude. For a beam with rectangular cross section, there will be an increase in each transverse oscillation amplitude by of the order of 20% for typical linac designs.

The foregoing analysis is subject to the following limitations:

1) We have used the smoothed approximation for the transverse oscillations. For typical parameters used in proton linacs, this should be valid to of the order of 10%.

2) We have included only terms up to the order x^4 in the magnetic scalar potential and have evaluated only those which give large contributions at the magnet ends.

3) We have approximated the fringing field by taking a "rectangular" model for the variation of quadrupole field with the axial coordinate. Our most serious error probably occurs here. It should be possible, however, to estimate this error from the numerical results of Regenstreif[1] for a variety of assumptions about the shape of the fringing field.

4) We have retained as the dominant term only that one containing the phase $2\phi_x - 2\phi_y$. In addition we have assumed a simple variation of parameters with velocity. The latter effect can easily be corrected for an actual design, whereas the former should be unimportant, except possibly for a small resonance at $\mu_x = \mu_y = 90°$.

References

1. E. Regenstreif, Los Alamos Linac Conference, October 1966, and private communication, p. 245.
2. E. D. Courant and H. S. Snyder, Ann. Phys. 3, 1-48 (1958).
3. R. Chasman, Los Alamos Linac Conference, October 1966, p. 224.

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