PHASE-SPACE REPRESENTATION OF ABERRATIONS DUE TO LONGITUDINAL FIELDS IN QUADRUPOLES

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Introduction

In a certain number of practical arrangements, the length of a quadrupole used for focusing purposes may be comparable to its aperture. This can happen for instance at the exit of a short pre-accelerating column or in the first focusing stages of a linear accelerator. We have tried to represent in phase space the situation at the exit of a focusing system where the presence of a longitudinal field and the coupling due to aberrations may influence the beam behavior.

A. Two-Dimensional Approach

We start from the general equations of motion

$$x'' = \frac{q}{mv}\,(1 + x'^2 + y'^2)\left[-(1 + x'^2)B_y + x'y'B_x + y'B_z\right]$$

$$y'' = \frac{q}{mv}\,(1 + x'^2 + y'^2)\left[(1 + y'^2)B_x - x'y'B_y - x'B_z\right]$$

and use the scaler magnetic potential

$$V = -G\left[xyk(z) + \frac{1}{12}\,(x^2 + y^2)\,xyk''(z)\right]$$

The third order equations of motion become then

$$x'' + k_o^2k(z)x = -k_o^2\left[k(z)\;x\;\frac{3x'^2 + y'^2}{2} - k(z)x'y'y - k'(z)xyy' - \frac{k''}{12}\,x(x^2 + 3y^2)\right]$$

$$y'' - k_o^2k(z)y = k_o^2\left[k(z)\;y\;\frac{3y'^2 + x'^2}{2} - k(z)x'y'x - k'(z)xyx' - \frac{k''}{12}\,y(3x^2 + y^2)\right]$$

In these equations $k_o^2 = \frac{qG}{mv}$, the other symbols are self-explanatory. The term in k' results from the force experienced by the particle when going through the longitudinal field B_z. The term in k'' translates the nonlinearity of the magnetic field in the fringing region.

If x_o, x'_o, y_o, y'_o describes the initial ray, the third order solutions of the equations of motion may be written in the form

$$x = ax_o + bx'_o + a_1x_o^3 + a_2x_oy_o^2 + a_3x_ox_o'^2 + a_4x_oy_o'^2 + a_5x_oy_oy'_o + a_6x_o^2x'_o + a_7y_o^2x'_o + a_8x_o'^3 + a_9x'_oy_o'^2 + a_{10}y_ox'_oy'_o$$

and similar expressions hold for x', y and y'. The 40 aberrations coefficients $a_1 \ldots a_{40}$ are related by 24 equations.

We have derived analytical expressions for the 40 coefficients for a variety of cases such as $k(z)$ = const. plus a δ kick, $k(z)$ represented by a bell-shaped curve, $k(z)$ represented by a straight line, etc. We have then written a program which enables us to calculate x,x',y,y' and the exit of any system composed of quadrupoles, taking into account aberrations and coupling. This method proves to be at least 100 times more rapid than the numerical integration of the equations of motion and also improves considerably the accuracy, particularly when a long system is considered.

Figures 1-5 show the results of the calculation as applied to a single quadrupole and to a triplet. The numerical values are those of a system actually used in the CERN-PS preinjection channel. In Fig. 1 we have represented the beam emittances at the entry of the system (quadrupole or triplet). Fig. 2 shows the beam behavior in the x,x' plane (focusing plane) at the exit of the quadrupole if one assumes for the characteristic function $k(z)$ a bell-shaped curve. Fig. 3 displays the situation in phase space in the case where $k(z)$ is a rectangle and the particle gets a δ kick at the ends of the quadrupole, equivalent to the action of the longitudinal field. Fig. 4 represents the situation in the cdc plane at the exit of a symmetric triplet the characteristics of which are indicated in the figure. In Fig. 5 the corresponding situation has been plotted in the dcd plane of the triplet. In all cases we have assumed that the starting ellipses (x_o,x'_o) and (y_o,y'_o) are independent and we have calculated all final points x,x' by associating to one point of the (x_o,x'_o) ellipse all points of the (y_o,y'_o) ellipse; the same applies to the y,y' plane.

A few remarks can be made in relation with the results obtained:

a) All points associated to a given initial point x_o, x'_o are very nearly on a straight line.

b) For a given initial point x_o, x'_o, the point of the (y_o, y'_o) ellipse which leads to the maximum displacements Δx and $\Delta x'$ is always the same.

c) If one traces out the curve corresponding to a given y_o, y'_o one obtains in the x, x' plane very nearly an ellipse; the system behaves therefore as a lens whose focusing properties vary as a function of the particular point y_o, y'_o which has been chosen.

We have emphasized these properties in the x, x' plane; obviously, similar properties hold in the y, y' plane.

d) The emittance of the outgoing beam is represented by the envelope of the preceding curves; it is seen that considerable beam blowup as well as distortion occurs in both planes.

e) The beam blowup in the case of the triplet does not seem to be essentially different from that of a single quadrupole; apparently there is at least a partial compensation of the aberrations inside a symmetric triplet.

Let us stress again the fact that the calculations apply to quadrupoles or to a system of quadrupoles in which the lengths of the elements are comparable to their diameters.

B. Four-Dimensional Approach

The two-dimensional representation shows non-conservation of phase-space area in the two basic planes. However, it is not realistic to represent coupling and the influence of aberrations in two planes which are supposed to be initially decoupled.

Obviously, the determination of the beam behavior at the exit of the system by means of the aberration coefficients and the basic correspondence from x_o, x'_o, y_o, y'_o to x, x', y, y' retain their validity, it is only the representation of the phase space properties which must be changed. Moreover, the density distribution of the particles in phase space should be taken into account.

In the four-dimensional approach we try to determine the domain D occupied by the beam at the exit and the density distribution in D as a function of the domain D_o occupied by the beam at the entry and the density distribution in D_o. We then consider the areas occupied by the beam in the x, x' and the y, y' planes as the projections onto these planes of the domain D in the x, x', y, y' space. Similarly, the cross section of the beam in a plane perpendicular to Oz is obtained by projecting D onto the x, y plane.

Consider then a beam occupying at the entry of the system a domain D_o and let this domain be bounded by a closed surface

$$\Psi(X_o, X'_o, Y_o, Y'_o) = 0.$$

All points in D_o can be described by

$$\Psi(X_o, X'_o, Y_o, Y'_o) \le 0.$$

Moreover, let $d(X_o, X'_o, Y_o, Y'_o)$ be the four-dimensional phase space density in the neighborhood of X_o, X'_o, Y_o, Y'_o.

To obtain the emittance of the beam in the plane X_o, X'_o for instance, we select a particular point $X_o = x_o$, $X'_o = x'_o$ and we examine the range inside which we can take x_o and x'_o; this range can be determined by writing that the inequality

$$\Psi(x_o, x'_o, Y_o, Y'_o) \le 0$$

must have solutions in Y_o and Y'_o.

The number of particles having excursion between x_o and $x_o + dX_o$ and slopes between x'_o and $x'_o + dX'_o$ is given by

$$dN = dX_o\, dX'_o \iint_{S_o} d(x_o, x'_o, Y_o, Y'_o)\, dY_o dY'_o$$

where the surface of integration S_o is determined by the condition

$$\Psi(x_o, x'_o, Y_o, Y'_o) \le 0$$

with x_o and x'_o fixed.

The two-dimensional phase space density at the point x_o, x'_o of the plane X_o, X'_o is therefore

$$\iint_{S_o} d(x_o, x'_o, Y_o, Y'_o)\, dY_o dY'_o.$$

A similar procedure may be used to calculate the area occupied by the beam in the Y, Y' plane at the exit of the system as well as the density in this plane; finally one can determine the cross section of the beam in a plane perpendicular to the propagation axis.

As a particular case, consider a beam of axial symmetry. With no more arbitrariness than in the two-dimensional approach (where the beam is represented by an ellipse in its phase plane) we may assume that the volume occupied in hyperspace by the incoming beam is represented by the hyper-ellipsoid

$$fX_o^2 + 2g\, X_o X'_o + hX'^2_o + fY_o^2 + 2g\, Y_o Y'^2_o + hY'^2_o - E = 0 \qquad (1)$$

In other words, all the particles of the beam will have at the entrance of system excursions X_o, Y_o and slopes X'_o, Y'_o so that

$$fX_o^2 + 2g\,X_oX'_o + hX_o'^2 + fY_o^2 + 2g\,Y_oY'_o + h\,Y_o'^2 - E \le 0.$$

The projection of the hyper-volume on the basic planes X_o, X'_o and Y_o, Y'_o are then the ellipses

$$fX_o^2 + 2g\,X_oX'_o + h\,X_o'^2 - E = 0$$

$$fY_o^2 + 2g\,Y_oY'_o + h\,Y_o'^2 - E = 0$$

The important point is now the following. Being given that the coordinates and slopes of any particle must satisfy Eq. (1), if one takes a point A on the boundary of the X_o, X'_o ellipse, the area of which is $\pi E/\sqrt{fh-g^2}$, the only point one can take in the Y_o, Y'_o to calculate the beam behavior is the point o,o (Fig. 6). If one takes a point B inside the X_o, X'_o ellipse and situated on a similar ellipse of smaller area $\pi C/\sqrt{fh-g^2}$ ($C < E$), one may associate to it in the plane Y_o, Y'_o all the points inside an ellipse similar to the limiting one and of area $\pi(E-C)/\sqrt{fh-g^2}$. Obviously, the same correspondence holds when going from the Y_o, Y'_o plane to the X_o, X'_o plane.

The basic independence and decoupling of the initial two-dimensional emittances disappear therefore in the four-dimensional approach.

At the entry of the system, the cross section of the beam in a plane perpendicular to the propagation axis may be obtained by projecting the hyper-ellipsoid onto the plane X_oY_o. To this effect we put $X_o = x_o$, $Y_o = y_o$ and examine the range inside which these quantities must be chosen in order that Eq. (1) admits solutions in X'_o, Y'_o. One should have

$$fx_o^2 + 2g\,x_o\,X'_o + h\,X_o'^2 + f\,y_o^2 + 2g\,y_oY_o'^2 - E \le 0$$

and this gives the condition

$$\left(X'_o + \frac{g}{h}x_o\right)^2 + \left(Y'_o + \frac{g}{h}y_o\right)^2 \le \frac{E}{h} - \left(x_o^2 + y_o^2\right)\left(\frac{f}{h} - \frac{g^2}{h^2}\right)$$

with x_o and y_o fixed.

The latter relation can only be satisfied if

$$x_o^2 + y_o^2 \le \frac{hE}{fh - g^2}$$

The cross section of the beam in a plane perpendicular to the propagation direction is therefore a circle of radius

$$R = \sqrt{\frac{hE}{fh - g^2}}$$

and the slopes of the trajectories which limit the beam at the entrance are given by

$$X'_o = -\frac{g}{h}X_o \qquad Y'_o = -\frac{g}{h}Y_o$$

At the exit of the system the hyper-volume representing the beam will no more be a hyper-ellipsoid if one takes into account aberrations and coupling. To calculate this volume we may use the correspondence.

$$X = X(X_o, X'_o, Y_o, Y'_o)$$

$$X = X'(X_o, X'_o, Y_o, Y'_o)$$

$$Y = Y(X_o, X'_o, Y_o, Y'_o)$$

$$Y = Y'(X_o, X'_o, Y_o, Y'_o)$$

which we can write out explicitly if we know the transfer matrices of the system and the aberration coefficients. We then solve this system for X_o, X'_o, Y_o, Y'_o.

$$X_o = X_o(X, X', Y, Y')$$

$$X'_o = X'_o(X, X', Y, Y')$$

$$Y_o = Y_o(X, X', Y, Y')$$

$$Y'_o = Y'_o(X, X', Y, Y')$$

and substitute these values in Eq. (1). We thus obtain an equation in X, X', Y, Y' which represents the limiting surface of the beam at the exit. The projections onto the planes X, X' and Y, Y' yield then the two-dimensional emittances which one looks for.

The numerical computations along these lines are under way and the results will be reported later.

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DISCUSSION

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FAURE, Saclay: At Saclay, from Prof. Regenstrief's studies, we have taken into account this kind of aberration for the transport line between the pre-injector and the linac. We were obliged to select quadrupoles the length of which are twice the diameter in order to keep the aberration factor of the order of a few percent.

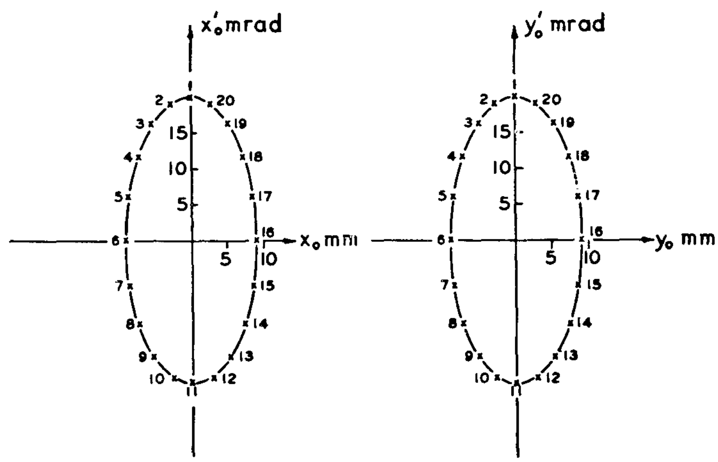


Fig. 1.

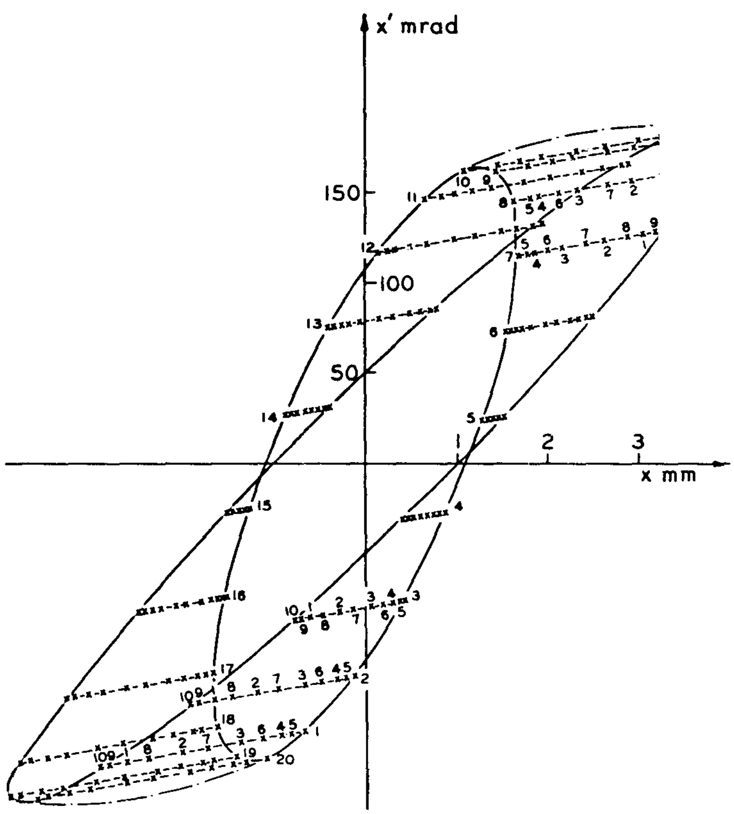


Fig. 2.

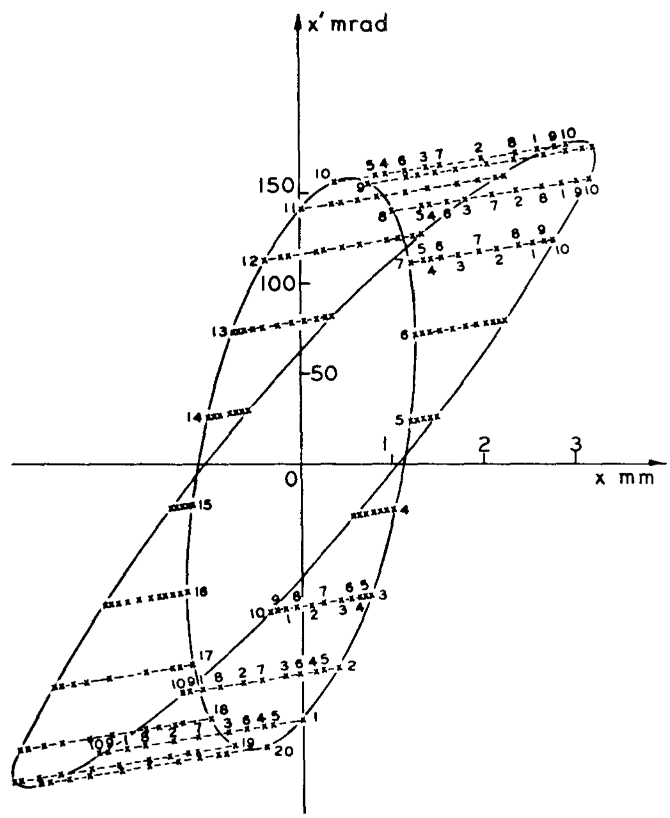
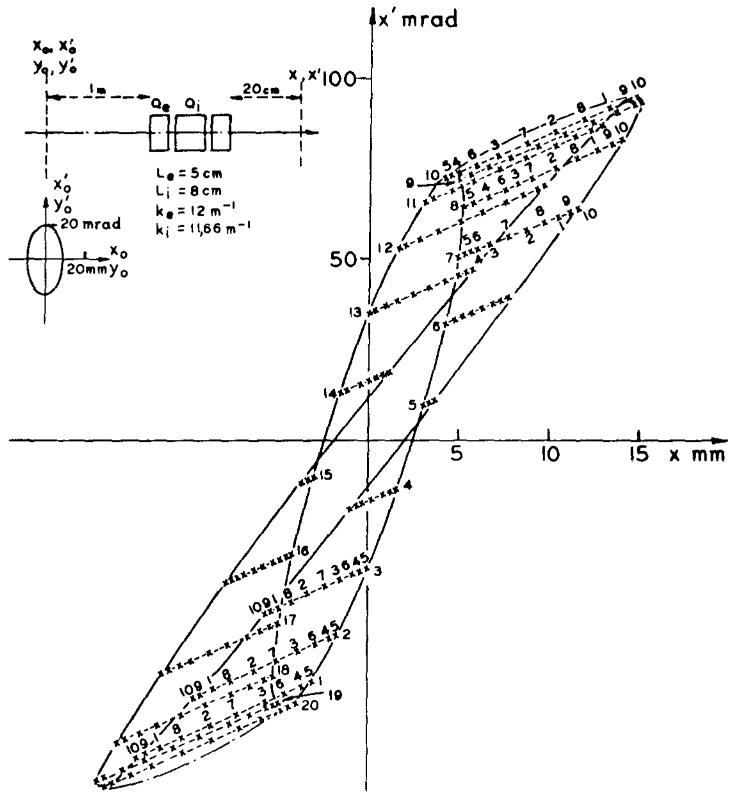


Fig. 3.



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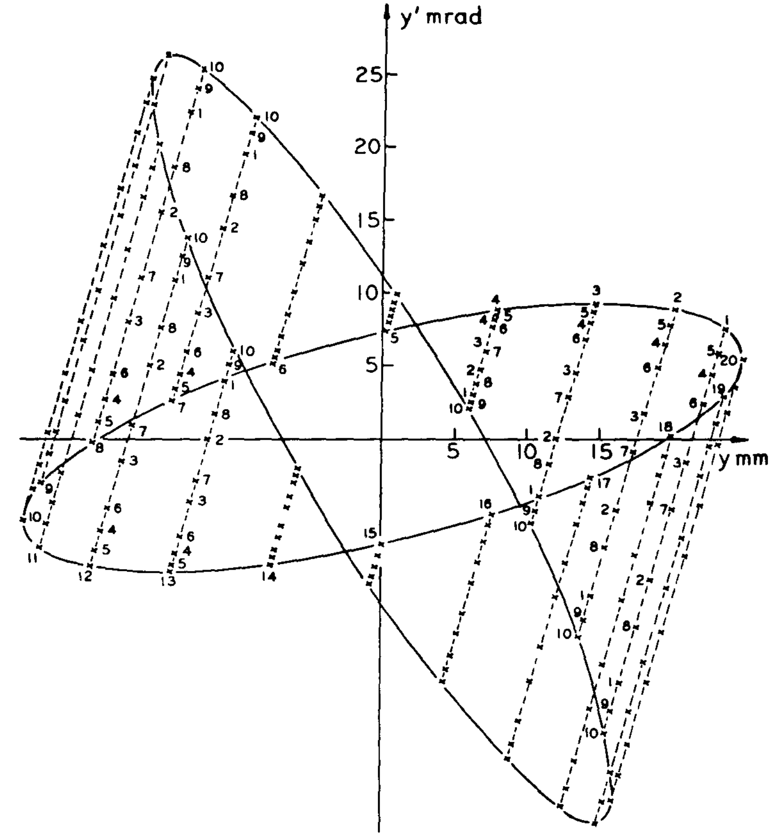

Fig. 5.

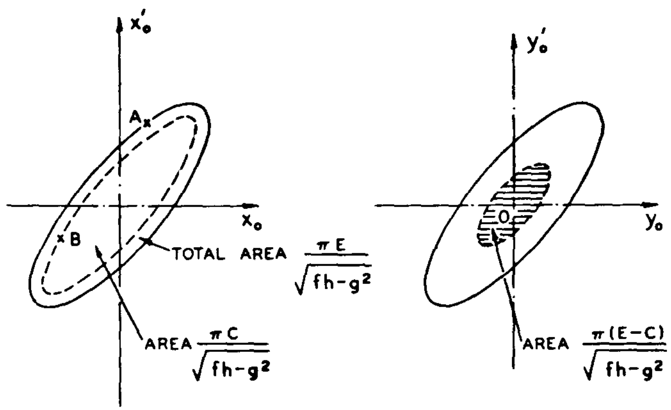

Fig. 6.