SPACE CHARGE EFFECTS IN PROTON LINEAR ACCELERATORS*

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I. Introduction

In recent years, the development of ion sources[1] has led to a substantial increase in the currents accelerated in proton linacs. Accelerated currents in excess of 100 mA have been obtained at CERN,[2] and similar currents are expected for the 200 MeV linac which is to be part of the AGS improvement program. The purpose of this note is to investigate the effects of high space charge on both the longitudinal and transverse motions in order to see whether these effects will be serious for the particular parameters of the planned linac.

The effect of space charge on the longitudinal oscillations has been considered previously.[3-5] Morton[3] has investigated the change in phase acceptance at injection due to the space charge forces and finds a substantial reduction of the longitudinal admittance for high currents. He does not, however, consider further effects as the beam is accelerated and damped longitudinally, thus increasing the relative importance of the space charge force to the focusing force. We shall include this effect here. Nishikawa[4] points out the approximate spherical geometry of the beam bunch and evaluates the change in the focusing potential and acceptance at injection, using a spherical bunch shape. Lapostolle[5] assumes a uniform beam bunch of ellipsoidal shape and presents formulas for the space charge forces, including image effects, which allow incorporating the space charge forces into both the transverse and longitudinal motion computational programs. We shall try to obtain rough analytical guides for the corresponding computational program under way at BNL.[6]

II. Effect of Space Charge on Longitudinal Motion

The (uncoupled) equation for the longitudinal motion, in the absence of space charge forces, is

$$\frac{1}{\beta^3\gamma^3}\frac{d}{ds}\left(\beta^3\gamma^3\frac{d\chi}{ds}\right) + \frac{k_\ell^2(\cos\phi - \cos\phi_s)}{\sin|\phi_s|} = 0 \tag{1}$$

We shall approximate this by expanding trigonometric functions up to and including 2nd order in their arguments, and by considering only non-relativistic motion, to obtain

$$\beta^{-3}\frac{d}{ds}\left(\beta^3\frac{d\chi}{ds}\right) + k_\ell^2\left(\chi - \frac{\chi^2}{2|\phi_s|}\right) = 0. \tag{2}$$

The present notation is:

ϕ_s synchronous phase ($\phi_s < 0$)

$\chi = \phi - \phi_s$

βc synchronous velocity

$k_\ell \simeq \left(\frac{2\pi e E_o T|\phi_s|}{mc^2\lambda\beta^3}\right)^{1/2}$, longitudinal oscillation wave number

$\delta\gamma = \gamma - \gamma_s = -\frac{\lambda}{2\pi}\beta^3\frac{d\chi}{ds}$

The stability limits can easily be obtained in the approximation of constant β by integrating Eq. (2) with respect to χ:

$$\left(\frac{2\pi}{\lambda\beta^3}\delta\gamma\right)^2 + k_\ell^2\left(\chi^2 - \frac{\chi^3}{3|\phi_s|}\right) = \text{constant}. \tag{3}$$

The "potential" corresponding to Eq. (3) is shown in Fig. 1 and indicates a phase stable region of "width"

$$\Delta\chi_{p.s.r.} \simeq 3|\phi_s| \tag{4}$$

and "height"

$$\Delta\gamma_{p.s.r} \simeq \frac{\lambda\beta^3 k_\ell|\phi_s|}{\pi\sqrt{3}} \tag{5}$$

Individual particles move in the potential of Fig. 1. The collection of these particles then produces space charge forces which must also be included in Eq. (1). We shall assume that the particles move symmetrically about $\chi = 0$. (This is only approximate for particles near the stability limit.) In addition, we shall assume with Lapostolle[5] that the physical beam bunch is an ellipsoid of uniform density for which we can write the interior fields in the absence of image effects. Specifically we have an interior potential which is[7]

$$V^{s.c.} = \frac{3(Q/\varepsilon_o)}{4\pi baa'}[-fz^2 - (\frac{1-f}{2})(x^2 + y^2)] \quad (6)$$

where b, a, a' are the semi-longitudinal and transverse axes of the ellipsoid and Q is the total charge in the bunch. The function f, of the parameter $b/\sqrt{aa'}$, is given by

$$f(\xi) = \begin{cases} \frac{1}{1-\xi^2}[1 - \frac{\xi \cos^{-1}\xi}{\sqrt{1-\xi^2}}], & \xi < 1 \\ \frac{1}{\xi^2 - 1}[\frac{\xi \cosh^{-1}\xi}{\sqrt{\xi^2 - 1}} - 1], & \xi > 1 \end{cases} \quad (7)$$

If one includes the longitudinal space charge force, Eq. (2) becomes

$$\beta^{-3}\frac{d}{ds}(\beta^3 \frac{d\chi}{ds}) + k_\ell^2[(1-\mu)\chi - \frac{\chi^2}{2|\phi_s|}] = 0, \quad (8)$$

where

$$\mu k_\ell^2 = \frac{3e\lambda If}{4\pi\varepsilon_o caa'b\, mc^2\beta^2} = \frac{90 \text{ ohms } eI}{mc^2}\,\frac{f\lambda}{aa'b\beta^2} \quad (9)$$

and where I is the beam current.

Equation (8) obviously can be obtained from Eq. (2) by the replacements

$$k_\ell \rightarrow k_\ell\sqrt{1-\mu} \quad (10)$$

$$|\phi_s| \rightarrow |\phi_s|\,(1-\mu), \quad (11)$$

where Eq. (10) is consistent with merely changing $|\phi_s|$ in the definition of k_ℓ, according to Eq. (11). The new stability limits, in the presence of space charge, are therefore

$$\Delta\chi^{s.c.}_{p.s.r.} \simeq 3|\phi_s|(1-\mu) \quad (12)$$

$$\Delta\gamma^{s.c.}_{p.s.r.} \simeq \frac{\lambda\beta^3 k_\ell|\phi_s|}{\pi\sqrt{3}}(1-\mu)^{3/2}. \quad (13)$$

These, however, are expected to be only approximate, since the description as an ellipsoid is not accurate for particles oscillating near the stability limit.

The importance of Eqs. (12) and (13) arises from the fact that μ can be expected to increase during acceleration. Specifically, if phase damping occurs, χ_{max} will decrease as $\beta^{-3/4}$, and b will therefore increase as $\beta^{1/4}$. Assuming constant a and a', one finds from Eq. (9), using the fact that $k_\ell \sim \beta^{-3/2}$, that

$$\mu \sim \beta^{3/4}. \quad (14)$$

One therefore expects to lose phase stability at some intermediate value of β, with the resultant longitudinal loss of the beam.

The situation is somewhat improved if one realizes that, when space charge forces become important, the longitudinal damping of χ_{max} no longer proceeds as $\beta^{-3/4}$. In fact χ_{max} approaches the "equilibrium" value for which $\mu = 1$. Nevertheless, the phase stable region continues to decrease in width while the bunch is no longer damped, and longitudinal stability is eventually lost. We shall now try to estimate at what energy this takes place.

One can treat the self-consistent problem involving the damping by looking at the linearized version of Eq. (8). The standard JWKB solution leads to a phase oscillation amplitude given by

$$\frac{\chi_{max}}{\chi_o} \simeq \frac{\beta_o^{3/2}}{\beta^{3/2}}\,\frac{k_{\ell o}^{1/2}}{k_\ell^{1/2}}\,(\frac{1-\mu_o}{1-\mu})^{1/4}$$

$$\simeq (\frac{\beta_o}{\beta})^{3/4}(\frac{1-\mu_o}{1-\mu})^{1/4} \quad (15)$$

where the subscript o refers to the value at the start of acceleration. From Eq. (9) one also has

$$\frac{\mu}{\mu_o} \sim \frac{\beta}{\beta_o}\frac{b_o}{b} = \left(\frac{x_{max}}{x_o}\right)^{-1} , \qquad (16)$$

since

$$\frac{b}{\beta\lambda} = \frac{x_{max}}{2\pi} . \qquad (17)$$

Setting $\xi = x_{max}/x_o$, one obtains from Eq. (15)

$$\xi\left(\frac{\xi - \mu_o}{1 - \mu_o}\right) = \left(\frac{\beta_o}{\beta}\right)^3 \qquad (18)$$

which describes the damping of the longitudinal oscillation as a function of β.

The width of phase stable region is given by

$$\frac{\Delta x_{p.s.r.}}{\Delta x^o_{p.s.r.}} = \frac{1 - \mu}{1 - \mu_o} = \frac{\xi - \mu_o}{\xi(1 - \mu_o)} . \qquad (19)$$

If one sets $x_o = |\phi_s|$ and $\Delta x^o_{p.s.r.} = \frac{3}{2}|\phi_s|$, the width of the phase stable region is equal to x_{max} where

$$\xi = \frac{3}{2}\frac{(\xi - \mu_o)}{\xi(1 - \mu_o)}, \quad \xi = \frac{3 - \sqrt{3 + 6(1-2\mu_o)^2}}{4(1 - \mu_o)} . \qquad (20)$$

This occurs at the corresponding value of β/β_o given by Eq. (18).

Figure 2 shows the dependence of the energy at which particles start to be lost longitudinally as a function of the single parameter μ_o. It is clear from the figure that the energy is a very sensitive function of μ_o.

III. Numerical Estimates for Existing and Planned Linacs

According to Eq. (9) and the definition of k_ℓ, the value of μ_o is given by

$$\mu_o \sim \frac{60 \text{ ohms } I}{E_o T \lambda} \frac{f(b_o/\sqrt{aa'})}{\phi_s^2} \frac{\lambda^2}{aa'} \qquad (21)$$

where we have taken $2b_o = \beta_o\lambda(3|\phi_s|/2\pi)$.

The numerical estimates are made difficult primarily by the uncertainty in aa'. Nevertheless we have used approximate values appropriate to advertised emittances to obtain values of μ_o. The parameters used and the results obtained are given in Table I.

The result obtained for the present AGS injector indicates that the effect is relatively unimportant. However, the higher current, smaller emittance, and higher gradient of the CERN linac suggest that particles should start being lost at about 3 MeV. The second of the CERN entries is for a beam cross section 3 times larger, corresponding to the observation[2] of an increase in beam emittance by a factor of 3 near the start of acceleration. Only a slight increase in gradient is necessary to raise the 35 MeV start of longitudinal loss to well beyond 50 MeV, so it is not surprising that such an effect has not been observed.

It also appears that the LASL linac will not exhibit difficulty in the 200 Mc sections, since the proposed current is only 20 mA. A proper relativistic treatment will be necessary to determine the seriousness of this effect in the 800 Mc section.

The results for the new BNL 200 MeV linac are somewhat disturbing however, and methods by reducing the seriousness of the effect are discussed in the next section.

IV. Methods to Reduce Effect of Space Charge on the Longitudinal Motion

The most obvious method of reducing the seriousness of the effect of space charge on the longitudinal motion is to relax the restriction on the transverse beam dimensions. This may take place to some extent naturally: Coupling of the longitudinal and transverse oscillations can cause an appreciable increase in the transverse beam emittance.[2,8] Magnet misalignments also lead to large transverse oscillation amplitudes, although only the incoherent part[9] will be effective in reducing space charge forces. One can further encourage the increase of transverse dimensions (without further increase of beam emittance) by designing the transverse focusing system to have increasingly larger β_{max} (smaller k_t) during the course of acceleration. However, the lower value of k_t will lead to larger bore requirements and more serious transverse emittance growths due to longitudinal-transverse coupling,[8] although this latter effect is less serious after the first few MeV of acceleration.

It may also be possible to increase k_ℓ^2 during acceleration. This can be accomplished by "tilting" the accelerating field gradient so as to make $|\phi_s|$ increase with β. One possible solution is to keep

$$k_\ell^2(1-\mu) = \frac{2\pi eE_oT}{mc^2\lambda\beta^3}|\phi_s| - \frac{90 \text{ ohms } eI}{mc^2}\frac{f\lambda}{aa'b\beta^2} \tag{22}$$

constant by arranging for $|\phi_s|$ to increase according to

$$|\phi_s| = |\phi_s^o| + \mu|\phi_s^o| \; . \tag{23}$$

This choice of $|\phi_s|$ keeps the width of the phase stable region constant and provides for continuous damping of the phase oscillation amplitude according to $\beta^{-3/4}$. However, μ then increases with β and the increased phase angle indicated by Eq. (23) rapidly becomes excessive. Moreover, larger values of $|\phi_s|$ require additional transverse focusing.

It therefore appears that some provision will have to be made for increasing the transverse amplitude (but not the transverse emittance) for acceleration of high currents well above 50 MeV.

These conclusions are presently being explored numerically at BNL.[4] A possible design to accomplish this is one in which β_{max} varies as $\beta^{3/2}$ leading to a transverse dimension given by

$$a = \sqrt{\frac{W}{\pi}}\,\frac{\beta_{max}}{\beta} \sim \beta^{1/4} \quad , \tag{24}$$

where W is the constant normalized emittance. With this variation of a and a', and a damping of χ_{max} as $\beta^{-3/4}$, one finds from Eq. (9) that

$$\mu \sim \beta^{1/4} \tag{25}$$

rather than proportional to β. The energy at which longitudinal spill out will occur will therefore increase greatly, since μ will only vary by a factor of order 2 during acceleration. The extent to which longitudinal space charge can be controlled by proper design of the transverse focusing system, and whether or not field tilting is helpful, is being explored numerically at BNL.[6]

V. Effect of Space Charge on Transverse Motion

The equation for the transverse motion, including space charge effects, is obtained from Eq. (6) in an analogous way to that for the longitudinal motion. Specifically one finds

$$\beta^{-1}\frac{d}{ds}\left(\beta\frac{dy}{ds}\right) + k_t^2(1-\bar{\mu})y = 0 \quad , \tag{26}$$

where

$$\bar{\mu}\, k_t^2 = \frac{45 \text{ ohms } eI}{mc^2}\frac{(1-f)\lambda}{aa'b\beta^2} \; . \tag{27}$$

Since the usual designs for constant transverse amplitude have k_t decreasing with β^{-1}, the value of $\bar{\mu}$ is approximately constant.

If $\bar{\mu} < 1$ at injection, no instability is expected to set in as it does for the longitudinal oscillation.

A convenient way of taking into account the transverse space charge is in terms of a $\Delta_{s.c.}$ equivalent to the Δ_{rf} defocusing term at the gap. If one defines Δ_{rf} so that

$$\Delta(\beta\lambda\, y') = \Delta_{rf}\, y \tag{28}$$

applied at a gap, one has for $\Delta_{s.c.}$

$$\Delta_{s.c.} = (\beta\lambda)^2\,\bar{\mu}k_t^2 = \frac{45 \text{ ohms } eI}{mc^2}\frac{(1-f)\lambda^3}{aa'b} \tag{29}$$

compared with

$$\Delta_{rf} = -\frac{\pi eE_oT\lambda}{mc^2\beta}\sin\phi \quad . \tag{30}$$

The contribution $\Delta_{s.c.}$ is approximately constant as β increases. One must therefore provide additional transverse focusing to overcome this defocusing force. Moreover, when matching the transverse emittance to the acceptance of the linac, one should use β_{max} and β_{min} calculated including space charge effects.

If one redesigns the transverse focusing system, as discussed in Section IV, so that the transverse beam dimensions increase as $\beta^{1/4}$, one finds from Eq. (27) that

$$\bar{\mu}\, k_t^2 \sim \beta^{-1/4} \quad . \tag{31}$$

However, k_t now decreases as $\beta^{-3/2}$, and therefore

$$\bar{\mu} \sim \beta^{1/4}$$

just as in Eq. (25) for the longitudinal space charge parameter. It is obvious that a further increase in the beam size obtained by arranging for k_t to decrease more rapidly will cause more serious effects for the transverse motion. Moreover more rapid decrease of k_t may cause increased misalignment and coupling effects.

VI. Summary and Conclusions

1. The effect of space charge on both the longitudinal and transverse motion is obtained non-relativistically on the assumption of an ellipsoidal beam bunch of uniform charge density.

2. Stability limits and damping of the longitudinal oscillation are derived as a function of beam current for a beam of constant transverse cross section. The effect of space charge is seen to increase as β increases, implying beam loss when the phase stable region becomes smaller than the damped beam. The energy at which this occurs is estimated for existing and planned linacs.

3. Redesign of the transverse focusing system to allow the transverse beam dimension to increase slowly with β is tentatively suggested. This should be further explored numerically.

4. The results obtained in this paper depend strongly on the assumption of an ellipsoidal beam. Numerical computations now in progress should give an indication as to the extent that these conclusions are valid for the usual output beam from a conventional gap buncher.

References

1. See, for example, A. van Steenbergen, IEEE Trans. Nuclear Science NS-12, No. 3, 746 (1965).
2. Private communication, C. Taylor, CERN.
3. P. L. Morton, Rev. Sci. Instr. 36, 1826 (1965).
4. Private communication, T. Nishikawa, University of Tokyo.
5. P. M. Lapostolle, CERN Report AR/Int. SG/65-15, July 15, 1965.
6. A. Benton, LASL Linac Conference, October 1966, p. 243.
7. We have assumed, with Lapostolle, an ellipsoid of revolution, and will approximate the longitudinal forces by taking the transverse radius to be $\sqrt{aa'}$. Exact fields can be obtained, but these are complicated and involve elliptic integrals (see, for example, Durand, Electrostatique, Les distributions, Masson 1964).
8. R. L. Gluckstern, LASL Linac Conference, October 1966; BNL Accelerator Dept. Internal Report AADD-120, September 1966.
9. R. L. Gluckstern, Los Alamos Internal Report MP-4/RLG-5, March 14, 1966.

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DISCUSSION

R. L. GLUCKSTERN, Massachusetts

VAN STEENBERGEN, BNL: Did you mean there is actually a tradeoff between the phase space projection in x-x′ and the projection in y-y′? You mentioned that if one becomes larger, the other becomes smaller.

GLUCKSTERN: No. For the collection of points we're talking about, there will be as many which will decrease in amplitude in x-x′, or in amplitude of oscillation, as will increase. Each phase space area will therefore appear to have grown. For rectangular beams the radius can grow, but for properly matched beams of circular cross section, the effect of the magnets will be primarily to rotate the beam.

BLEWETT, BNL: I have the impression that the experimental measurements on low-energy sections of linacs indicate a blowup of emittance of something like a factor of 2 or 3. Am I correct in understanding you that we don't really have an explanation for an effect as big as that?

GLUCKSTERN: I'm not sure that statement is correct. Rena Chasman will talk about the numerical magnitudes of both the formulas and the corresponding calculations which are done. They indicate, in some cases, on the order of 30 or 40% changes in amplitude, which would correspond to factors of approximately 2 in phase space area. Now there are undoubtedly a lot of other things going on which are not really understood. One of these, space charge, I'll say a few words about later on in the morning. But I think the effects presented above are major contributors to the growth and transverse amplitude. I would agree with you, however, that this is not the full explanation for the beam growth.

MILLS, MURA: In general, for coupling resonances of the type considered in your paper, there exist approximate constants of the motion which are linear combinations of the adiabatic invariants of each degree of freedom. For difference resonances, the sum of the adiabatic invariants is constant; for sum resonances, the difference is constant.

GLUCKSTERN: Yes, I agree. This was pointed out to me by Ernest Courant at an earlier stage.

TABLE I

Linac	β_o	$E_o T \lambda$	I	aa'	b_o	f	μ_o	Energy
AGS	0.040	2.0 MeV	50 mA	0.25 cm^2	0.75 cm	0.23	0.12	> 200 MeV
CERN	0.033	3.0 MeV	125 mA	0.06 cm^2	0.6	0.14	0.5	~ 3 MeV
CERN	0.033	3.0 MeV	125 mA	0.2 cm^2	0.6	0.25	0.28	~ 30 MeV
New BNL	0.040	1.5 MeV	100 mA	0.1 cm^2	0.75 cm	0.14	0.5	~ 5 MeV
New LASL	0.040	1.5 MeV	20 mA	0.1 cm^2	0.75 cm	0.14	0.1	> 200 MeV

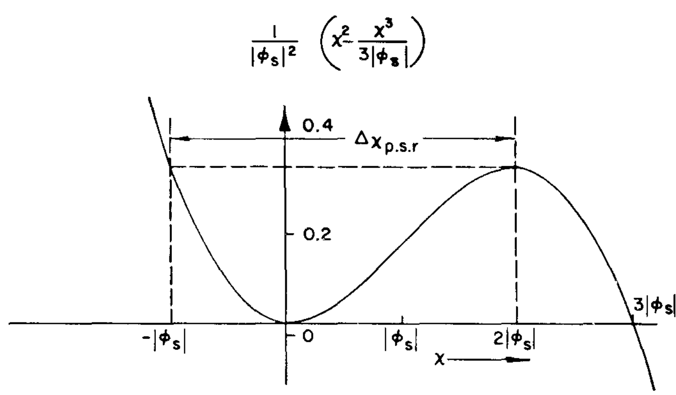

Fig. 1. "Potential" for longitudinal motion vs phase.

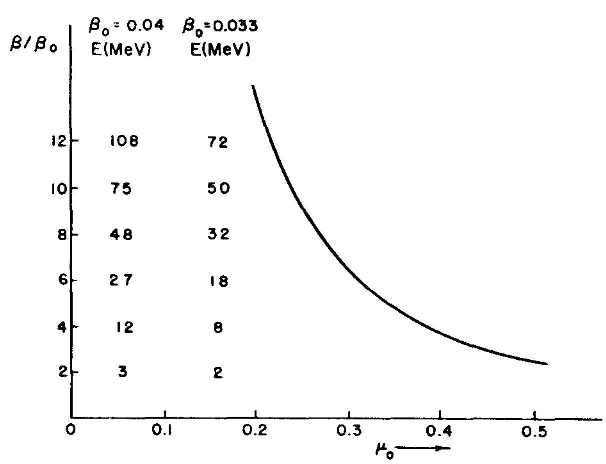

Fig. 2. Energy at which particles start being lost vs space charge parameter μ_o.