A NUMERICAL EXPERIMENT ON SPACE-CHARGE EFFECTS*

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Introduction

Kapchinskij and Vladimirskij,[1] and also Walsh,[2] obtained differential equations relating the radius of a charged-particle beam to the emittance, space-charge forces, linear focusing forces, and accelerating forces. However, the assumptions necessary for deriving these beam profile equations were that the charge density was uniform and the phase-space distribution was such as to maintain a uniform charge density. This required distribution confines particles to the surface of an ellipsoid in phase space. For an axially symmetric beam, this means that the magnitude of the transverse velocity of a particle at a given radius is restricted to the maximum value allowed by the ellipsoid. It would seem that a more realistic phase-space distribution would be one in which the particles could lie anywhere within an ellipsoid in phase space. This allows the magnitude of the transverse velocity of a particle at a given radius to have any value from zero up to the maximum one allowed by the ellipsoid. Since this latter phase-space distribution does not preserve a uniform charge density, several questions arise: How does the charge density change? How does a nonuniform charge density affect the apparent emittance? Under what conditions, if any, does the beam profile differ appreciably from that predicted by the beam profile equation?

In an effort to get a feeling for the answers to these questions, a numerical experiment was devised to study the effects of space charge on the charge density, apparent emittance, and the beam profile of an axially symmetric, non-accelerating beam. All of the particles in the beam are assumed to have the same axial velocity $\dot{z}$, and the beam radius and the charge density are assumed to change slowly enough with respect to z so that the electric field due to space charge is entirely in the radial direction.

Numerical Procedure

Since the beam is axially symmetric, it is convenient to think of the beam cross section at some particular point along the z-axis as being made up of a large number of equally-charged circular rings. That is, if there are N rings and the total current in the beam is I, then each ring carries a current of I/N. The space-charge forces being radial, all of the particles that are initially on any given ring will stay together on a circular ring. Each of the rings can shrink or grow in size as it moves along the axis, and can also be rotating about the axis. The initial radius, r, of each ring is dictated by a specified initial charge density. r' and $r\theta'$, the rates at which each ring is initially growing and rotating, are prescribed by a specified initial distribution in phase space. As these rings move together along the axis, they will in general be crossing, and the charge density and the phase-space distribution can be changing.

The radial force on a particle on any given ring depends simply on the ring radius and on the amount of charge inside the ring. If an index is assigned to each ring according to size, so that $o < r_1 < r_2 < \ldots < r_N$, then the radial force on a particle of charge e on the jth ring is

$$F_r(r_j) = e(j-\tfrac{1}{2})I/2\pi\,\epsilon_o N\,\dot{z}\,r_j.$$

Instead of following the r' and $r\theta'$ of each ring as it moves along the axis, it turns out to be easier to choose a particle at random from each ring and to follow the x, y, x', and y' coordinates of these particles, using the x and y components of the radial force. Given the values of x_j, y_j, x_j', y_j', and r_j for the jth ring at some point z along the axis, their values at $z + \Delta z$ may be found using the following transformations:

$$x_j(z+\Delta z) = x_j(z) + \Delta z\, x_j'(z) + \frac{(\Delta z)^2}{2}\, x_j''(z)$$

$$y_j(z+\Delta z) = y_j(z) + \Delta z\, y_j'(z) + \frac{(\Delta z)^2}{2}\, y_j''(z)$$

$$r_j^2(z+\Delta z) = x_j^2(z+\Delta z) + y_j^2(z+\Delta z)$$

$$x_j'(z+\Delta z) = x_j'(z) + \frac{\Delta z}{2}\left[x_j''(z) + x_j''(z+\Delta z)\right]$$

$$y_j'(z+\Delta z) = y_j'(z) + \frac{\Delta z}{2}\left[y_j''(z) + y_j''(z+\Delta z)\right]$$

Since some rings will, in general, cross during a step of Δz, the values of $x_j''(z+\Delta z)$ and $y_j''(z+\Delta z)$ are unknown at the beginning of a step; therefore, $x_j'(z+\Delta z)$ and $y_j'(z+\Delta z)$ cannot be completely calculated at the beginning of a step. However, the transformations of x_j, y_j, and r_j, and the first two terms in the transformations x_j' and y_j' can be computed. After this has been done for all N rings, the rings can be re-indexed according to size, at which time the last terms in the transformations of x_j' and y_j' can be computed.

This transformation of the coordinates is continued for a specified number of steps, at which time the charge density and the phase-space distribution are plotted. The charge density is computed by dividing the beam radius into a number of intervals, counting the rings in each interval,

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and dividing by the appropriate area. Each ring may be represented in x-x′ space by plotting the (x,x′) coordinates of four particles 90° apart. For the jth ring, these coordinates are (x_j,x_j'), (y_j,y_j'), $(-x_j,-x_j')$, and $(-y_j,-y_j')$. The beam profile can also be plotted by saving at every step the radius of the largest ring.

The effects of ideal focusing lenses may be included at various points along the axis by finding the r' and $r\theta'$ coordinates of each ring, changing r' to $r'- Ar$, where A is a specified positive constant depending on the focal length of the lens, and then transforming back to the x,y,x′ and y′ coordinates of the particles being followed.

Results

The results of several cases are shown in Figs. 1 through 7. In all cases, the beam consisted of 0.75 MeV protons, had an initial waist of 1 cm, and carried a current of 50 ma. The beam was allowed to approximately double in size, at which point was placed the first of three ideal focusing lenses. Both the position and strength of this lens were determined by the beam profile equation,

$$\frac{d^2R}{dz^2} = \frac{eI}{2\pi\epsilon_o m\dot{z}^3 R} + \frac{(E/\pi)^2}{R^3} ,$$

such that the beam, if it obeyed this equation, would be focused to a 1 cm waist, the lens being equidistant from both waists. The second and third lenses were identical to the first, and positioned so that the same beam profile should be repeated twice more.

Figure 1 shows the results of a 5π mrad-cm emittance beam having an initial uniform charge density, and for which the transverse velocities were chosen from the surface of an ellipsoid in phase space. The results are in agreement with theory: the beam profile equation was obeyed; the charge density remained relatively uniform; there was no warping of phase space. This was one check on the accuracy of the numerical procedure. Another check was performed as follows: The rings were followed a certain distance until the beam profile approximately doubled its initial size, at which point the sign of the r' coordinate of each ring was reversed and the rings followed an equal distance to see if the initial conditions would be satisfactorily reproduced. They were in all cases.

In the remainder of the cases, the transverse velocities were chosen from the interior of the phase-space ellipsoid. Starting with a uniform charge density, in a beam with a small emittance (1 π mrad-cm) the charge density tends to remain uniform, and the beam profile agrees with that given by the beam profile equation. For larger emittances (5 π or 10 π mrad-cm), the charge density tends to become higher at the center of the beam, and in one case seems to oscillate between a uniform charge density and a high density in the center. The nonuniform charge density cases a warping of the phase-space ellipse so that, at a waist, it looks more like a rectangle or a bow tie. In addition, the beam profile differs somewhat from that predicted by the beam profile equation.

When the initial charge density is gaussian, even small emittance beams are distorted in phase space. In the zero emittance beam shown in Fig. 7, the charge density is seen to alternate between a high density in the center and a high density at the beam edge.

In addition to the results shown in the figures, a movie was generated with the computer for the cases shown in Figs. 1 through 6. The movie shows the motion of the particles in x-x′ space, along with the trajectory of the beam profile.

References

1. I. M. Kapchinskij and V. V. Vladimirskij, "Limitations of Proton Beam Current in a Strong Focusing Linear Accelerator Associated with the Beam Space Charge," Conference on High-Energy Accelerators and Instrumentation, C.E.R.N. Geneva (1959).
2. T. R. Walsh, "A Normal Beam with Linear Focusing and Space-Charge Forces," Journal of Nuclear Energy, Part C, Vol. 5, pp. 17-22, (1963).

DISCUSSION

K. R. CRANDALL, LASL

BLEWETT, BNL: There is another phenomenon which often happens in beams of this sort and which has been looked into by Clarence Turner at Brookhaven. (He has written a report on the subject.) On its way to 750 keV the beam often has picked up a core of electrons from ionization of the residual gas. This has quite violent effects, the primary one of which is that the beam begins to appear hollow.

LAPOSTOLLE, CERN: I would like to ask you whether, in this process, there is an apparent phase space growth?

CRANDALL: Yes, because the ellipses occupied by the particles in x-x′ space were distorted.

VAN STEENBERGEN, BNL: I thought you showed one case whereby you started out with zero emittance and showed a finite emittance - at least your picture seemed to indicate a finite emittance - or were you indicating line density?

CRANDALL: In that particular case, the area of the x-x′ projection did increase from zero to some finite value. However, if the particle had been projected onto the r-r′ plane, then you would not have seen any area. The particles would initially lie on a straight line, and at a later time would lie on a curved line on the r-r′ plane.

SWENSON, LASL: I'm sure if he were to plot the resulting situation in four-dimensional phase space, you'd find that it still has zero volume. But the projection on x-x′ does have an area, whereas the projection on the r-r′ phase space does not show any area.

SEPTIER, Orsay: How many points do you calculate in each phase space ellipse?

CRANDALL: I usually use from 1000 to 2000 rings to simulate the beam cross-section, and each phase-space ellipse consists of 4 points per ring.

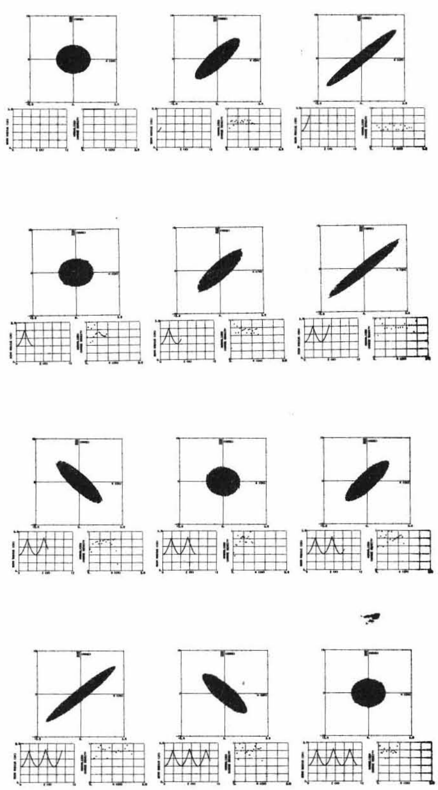

Fig. 1. Initial uniform charge density; E = 5π mrad-cm; transverse velocities chosen from ellipsoid surface.

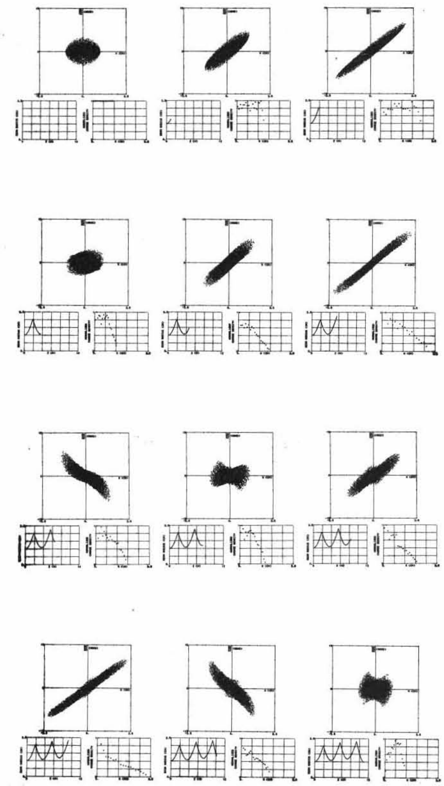

Fig. 2. Initial uniform charge density; E = 5π mrad-cm; transverse velocities chosen from ellipsoid interior.

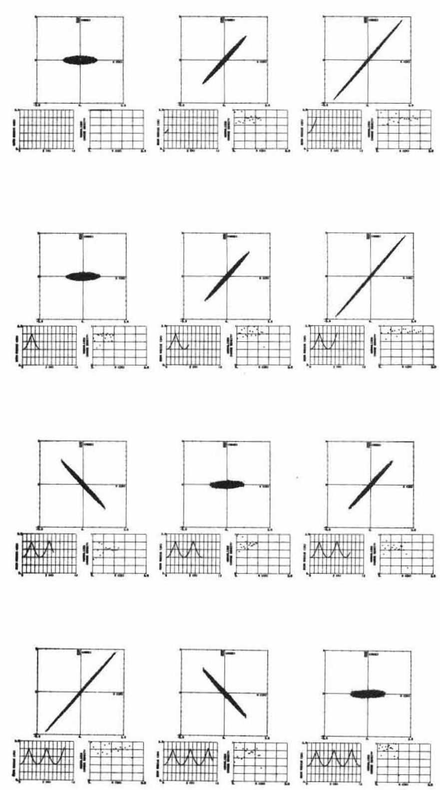

Fig. 3. Initial uniform charge density; E = 1π mrad-cm; transverse velocities chosen from ellipsoid interior.

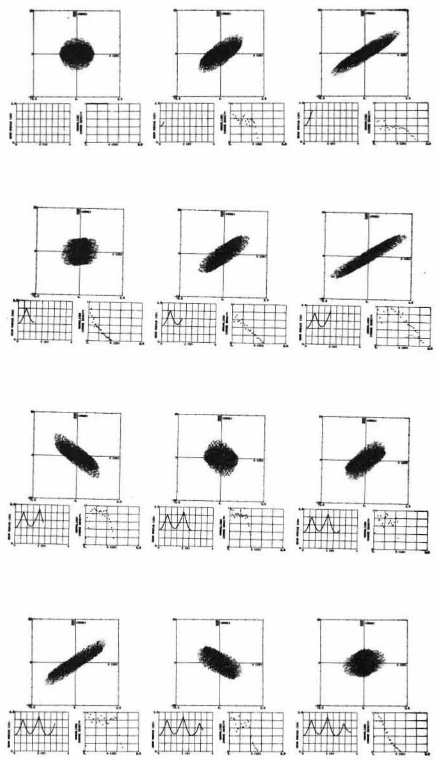

Fig. 4. Initial uniform charge density; E = 10π mrad-cm; transverse velocities chosen from ellipsoid interior.

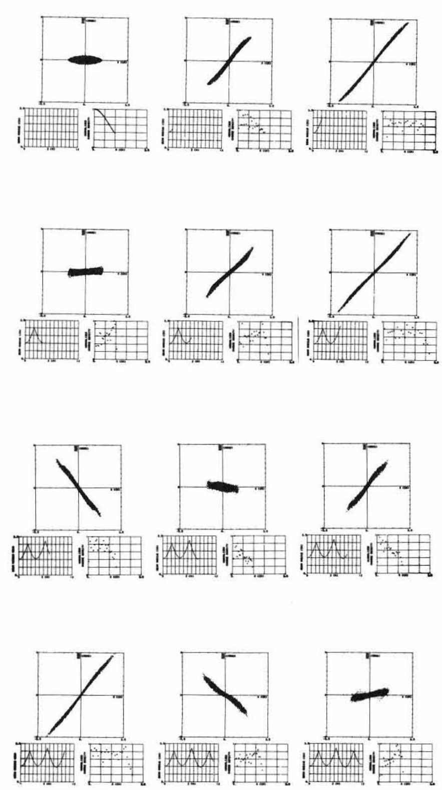

Fig. 5. Initial Gaussian charge density; E = 1π mrad-cm; transverse velocities chosen from ellipsoid interior.

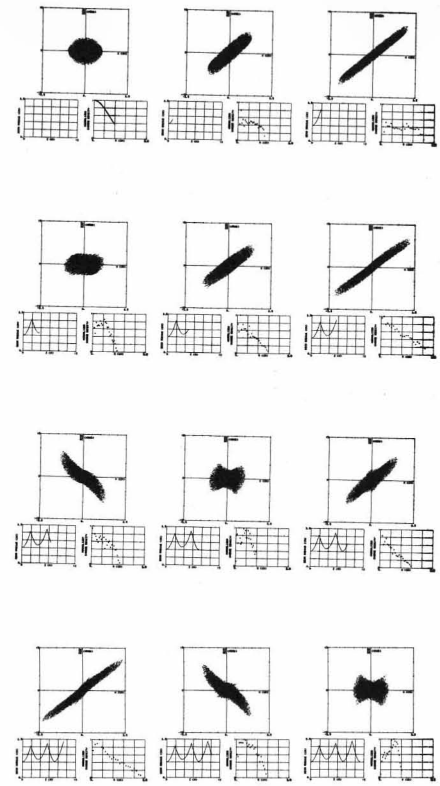

Fig. 6. Initial Gaussian charge density; E = 5π mrad-cm; transverse velocities chosen from ellipsoid interior.

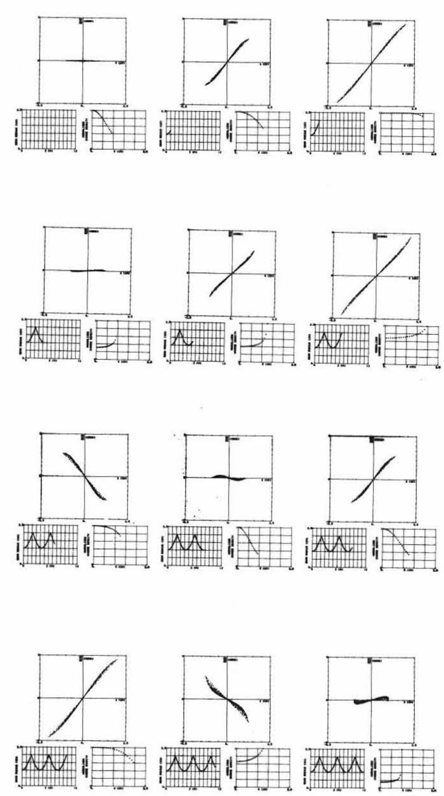

Fig. 7. Initial Gaussian charge density; E = 0.