NUMERICAL CALCULATIONS OF COUPLING EFFECTS IN A LOW ENERGY PROTON LINAC*

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Introduction

The following is a brief description of numerical calculations on the coupling effects mentioned by R.L. Gluckstern in preceding papers,[1,2] namely coupling between the longitudinal and transverse motions and coupling between the two transverse motions due to the quadrupole fringing fields. In the former case, only the influence on the transverse motion was considered.

All computations were done on the CDC 6600 computer at Brookhaven National Laboratory. The basic particle dynamics program used was developed by S. Ohnuma and collaborators at Yale. It assumes that the acceleration takes place in an infinitesimally small interval. The transit time factor in each cell is a function of the energy of each individual particle and a change in phase is introduced at the accelerator gap to guarantee conservation of longitudinal phase space area (Promé approximation).

Most of the machine parameters used were present design values of the 200 MeV injector linac for the AGS conversion program. Only + - + - quadrupole configurations were considered. No space charge effects were included.

Coupling Between Longitudinal and Transverse Motions

Computer runs were performed on the emittance in a transverse plane for beams with different initial longitudinal phases and amplitudes. Figure 1 shows the emittances at 10 MeV for five different initial values of longitudinal coordinates. The units are chosen such that the initial emittance is a circle and remains a circle (to first approximation) for the synchronous particle. As predicted by Gluckstern the common circle transforms into ellipses of different orientation. Each of these ellipses rotates going down the machine and consequently there will be an increase in the transverse amplitude and an apparent increase in the transverse phase space area. The area of each ellipse equals the area of the original circle to a first approximation. The final fractional apparent increase in transverse phase space area will be twice the final maximum fractional increase in transverse amplitude, which is given by Gluckstern as**

$$\delta_{max}(\infty) = \frac{(R_{max})_{max} - R_i}{R_i} = \frac{\chi_o g k_\ell^2}{8k_t}\left[\frac{1}{|2k_t - k_\ell|} + \frac{1}{2k_t + k_\ell}\right] + \frac{\chi_o^2 k_\ell^2}{32\, k_t}\left|\frac{g^2\left(\frac{1}{3} + \frac{k_\ell}{2k_t - k_\ell}\right) - 1}{2k_t - 2k_\ell}\right| .$$

Figure 2 shows $\delta_{max}(s)$, i.e. δ_{max} as function of distance down the first part of the linac (up to 10 MeV), for two different values of θ^2 (or initial magnetic field gradients). θ^2 is taken constant over the entire energy range. It can easily be recognized that δ_{max} settles down to a nearly constant value after a few MeV and that no local value of δ_{max} is larger than twice the asymptotic value at 10 MeV as stated in Ref. 1.

Figure 3 shows δ_{max} at 10 MeV as a function of θ^2 for different values of χ_o, the initial longitudinal amplitude. The first or dominant term in the analytical expression for δ_{max} is also shown as a function of θ^2 for $\chi_o = 0.45$. One can see that this then accounts for a large part of δ_{max}. Furthermore, in support of the same fact, δ_{max} is nearly proportional to χ_o and is zero for synchronous conditions. Similar information can be obtained from Fig. 4. Here, δ at 10 MeV is plotted as a function of the initial longitudinal phase, a_ℓ. According to Ref. 1, $\delta(\infty, a_\ell, a_t)$ is given analytically as:

$$\delta(\infty, a_\ell, a_t) = \frac{g\, k_\ell^2}{8\, k_t}\chi_o\left[\frac{\sin(2a_t - a_\ell)}{2k_t - k_\ell} - \frac{\sin(2a_t + a_\ell)}{2k_t + k_\ell}\right] + \frac{k_\ell^2 \chi_o^2}{32\, k_t}\frac{\cos(2a_t - 2a_\ell)}{2k_t - 2k_\ell}\left[1 - g^2\left(\frac{1}{3} + \frac{k_\ell}{2k_t - k_\ell}\right)\right]$$

This expression has two maxima as a_ℓ goes from 0° to 360° all values of a_t being considered. If only the first dominant term would be present these two maxima would be of the same height. This is nearly the case for $\theta^2 = 0.50$ and $\theta^2 = 0.45$, while for $\theta^2 = 0.425$ and 0.40, the two maxima differ considerably. It was for these last two values of θ^2 that the largest discrepancy was found between the computed δ_{max} and the dominant term in its analytical expression (see Figure 3)

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**For notation, see Ref. 1. δ is called $\delta A/A$ Ref. 1.

$\delta_{max}^{(\infty)}$ was also computed for two different initial transverse amplitudes, namely 1 and 0.5 cm respectively. All other parameters were kept constant. The values obtained were within 3% of each other. This is in good agreement with the analytical expression for $\delta_{max}^{(\infty)}$ which predicts no dependence on the initial transverse amplitude.

As has been pointed out by Gluckstern, only moderate agreement with orbit computations was obtained for the second term of $\delta_{max}^{(\infty)}$ and $\delta(\infty, a_\ell)$. The latter comparison was done by Fourier analysis. In all cases, this second term comes out smaller in the orbit computation than in the analytical calculation, including the case where initially $k_t = k_\ell$, which occurs for $\theta^2 = 0.475$.

It was earlier mentioned that the emittance in transverse phase space corresponding to an initially single point in longitudinal phase space remains constant to a first approximation, i.e. a circle transforms into an ellipse of equal area. This was in fact found not to be accurately true, but the transverse phase space area turned out mostly to decrease, locally and asymptotically while there was a corresponding increase in longitudinal phase space area, in agreement with Poincare's theorem. Figure 5 shows the fractional change in actual transverse and longitudinal phase space areas, $\Delta_t(s)$ and $\Delta_\ell(s)$ up to 10 MeV. For $\theta^2 = 0.40$ a local decrease of 30% in transverse phase space area is observed. At 10 MeV this decrease has settled down to 20%. The upper line shows that the sum of the transverse and longitudinal phase space areas is constant within the limits of computation errors.

Gluckstern gives the following analytical expression* for the fractional change in transverse phase space area:

$$\Delta_t(\infty, a_\ell) = \frac{A_t(\infty) - A_i}{A_i} = -\frac{k_\ell^3}{32\,k_t}\left(\frac{\pi y_o}{\beta\gamma\lambda}\right)^2 \left\{ \left| \frac{g}{(2k_t - k_\ell)} - \frac{i\,\chi_o\,e^{-ia_\ell}}{2k_t - 2k_\ell}\left(\frac{g^2\,k_\ell(k_\ell + k_t)}{4k_t^2 - k_\ell^2} - \frac{1}{2}\right) \right|^2 - \left(\frac{g}{2k_t + k_\ell}\right)^2 \right\}.$$

Figure 6 shows $\Delta_t(a_\ell)$ at 10 MeV. Again, the dominant (first) term in the analytical expression of $\Delta_t(\infty, a_\ell)$ is also shown on the graph. It is independent of a_ℓ.

The second term has a sinusoidal dependence on a_ℓ while the third term is constant. As one can see from Fig. 6, the values of Δ_t obtained from orbit computations show approximately the predicted dependence on a_ℓ .

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* Δ_t is called $\dfrac{\delta\text{Area}^{(t)}}{\text{Area}^{(t)}}$ in Ref. 1.

x-y Coupling Due to Quadrupole Fringing Fields

The fringing field of the quadrupole magnets causes additional forces in the x and y directions, which depend both on x as well as on y coordinates. Consequently there will be coupling between the two transverse motions. This phenomena was analyzed in detail by E. Regenstreif[3] for transport systems and by R.L. Gluckstern[2] for linacs.

In order to do linac orbit computations showing the influence of the quadrupole fringing fields on the transverse motion, it was assumed that the scalar potential of the quadrupole magnetic field is constant inside each magnet and zero elsewhere (rectangular model). Particles will then get impulses in the x and y directions at the entrance and exit of each magnet. The analytical forms of these impulses have already been given in Ref. 2 as formulas 12 and 13. The same expressions were used to modify the particle dynamics program so that it would include the effects of quadrupole fringing fields.

Figure 7 shows the emittance in the $x - p_x$ plane at 10 MeV for various initial values of y and p_y. A fractional increase of 18% in the maximum x-amplitude is obtained. This compares well with 22.5% obtained from the analytical expression in Ref. 2:

$$\left.\frac{\Delta A_x}{A_x}\right|_{max} \simeq \frac{3}{16}\,\beta_1\left(\frac{\theta^4}{\ell^3}\right)A_y^2\left(\frac{\beta_{inj}}{2\Delta\beta}\right)$$

with parameters used in the orbit computation. Similar results were obtained for the fractional increase in maximum y-amplitude.

As pointed out by Gluckstern, increases in maximum x and y amplitudes to first approximation, do not cause an increase in the maximum radial amplitude. The reason is that for an individual particle the change in the emittance invariants W_x and W_y will be of the same magnitude but of opposite sign every time the particle passes a magnet. The analytical forms for the changes in emittance invariants are given in (28) and (29) in Ref. 2 by:

$$\Delta W_x \simeq A_x^2\,A_y^2\,\sin(2a_x - 2a_y)\left(\frac{3}{8}\frac{\beta_1}{\beta_o}\right)\beta\gamma\,K^2\,\ell$$

$$\Delta W_y \simeq A_x^2\,A_y^2\,\sin(2a_y - 2a_x)\left(\frac{3}{8}\frac{\beta_1}{\beta_o}\right)\beta\gamma\,K^2\,\ell$$

Consequently, $\Delta R^2 \propto \Delta W_x + \Delta W_y = 0$.

Figure 8 shows the equal decrease and increase in x and y amplitudes respectively for a particle with $2a_y - 2a_x = \pi/2$. The uncoupled x (or y) oscillations are also shown.

Higher order terms in ΔW_x and ΔW_y will, however, cause a small increase in radial amplitude. Gluckstern predicts that this increase will be largest for a phase advance, μ, of 90^o per magnet period. Such a small resonance effect has indeed been observed in orbit computations. Table 1 gives the maximum fractional increase in radial amplitude at 10 MeV for various values of μ.

Table 1

θ^2	μ	$\frac{\Delta R_{max}}{R_i}$
0.8	77.4^o	0.04
0.9	90.5^o	0.06
1.0	105.4^o	0.04

The magnetic field gradients corresponding to these values of μ are, however, unrealistically high. A phase advance of 90^o corresponds to an initial field gradient of 12.5 kg/cm for other conventional machine parameters.

The agreement achieved between computer calculations and relatively simple analytical formulas is encouraging. Similar work will soon be performed on other important effects, such as space charge and misalignment errors. Hopefully, this will lead to equally good results so that the computer work necessary for designing a linac will be reduced.

References

1. R.L. Gluckstern, Los Alamos Linac Conference, October 1966 and BNL Accelerator Dept. Int. Rpt. AADD-120, **(these proceedings, p. 207.)**

2. R.L. Gluckstern, Los Alamos Linac Conference, October 1966 and BNL Accelerator Dept. Int. Rpt. AADD-122, **(these proceedings, p. 237.)**

3. E. Regenstreif, Los Alamos Linac Conference, October 1966, and private communication, **(these proceedings, p. 245.)**

DISCUSSION

R. CHASMAN, BNL

LAPOSTOLLE, CERN: It is clear from what you showed us that the focusing strength is very important. How do you adjust it during acceleration?

CHASMAN: I should have pointed that out. In computer runs I have kept θ constant clear through the first tank, which means that the gradient goes down.

GLUCKSTERN: I wanted to point out a few things which I neglected to mention or other things which might be relevant to this talk. The manner in which the formulas are obtained for the amplitude increases come about primarily from accumulation of damped oscillatory terms. As a result, if you take the integral of a damped oscillation, you find intermediate peaks which come close to twice the final value. In trying to assess the seriousness of the growth at intermediate values of longitudinal distance, one should use as an upper limit twice the final amplitude. However, if there are several terms, they all will not reach the double value at the same energy. The factor two, therefore, really represents an upper limit.

Another thing I wanted to mention has to do with the $2k_t - 2k_1 = 0$ resonance. The formulas are not valid if k_t is equal to k_1 during acceleration. In that case one has to use a different formalism, in which one assumes a linear variation of the difference between the frequencies and gets Bessel functions of order 1/3. This means that if k_t and k_1 are near to one another at the start of acceleration or cross one another during acceleration, that one shouldn't expect to be able to get accurate numerical agreement between the computations and the formulas that were given. The fact that the term enters the numerical results, however, is clear from one of the slides which was just shown, namely the one with the double peaks. For those low values of θ^2, the two peaks were of different size, and this can only come from a term of the type $2k_t - 2k_1$.

The last thing I wanted to mention has to do with the x-y coupling. There was a term $\alpha''(s)$, which enters as a correction to the H_x and H_y. I believe in some previous calculations which use rectangular models, the α'' term, which turns out to be the second derivative of the delta function, was omitted.

CHASMAN: It was accounted for here.

PROMÉ, Saclay: I just want to mention that we have made some computations at Saclay, taking into account the fringing field of the quadrupoles, and that we have not observed any increase in the radial phase space area due to the fringing field, especially when you take into account the hyper-ellipsoidal distribution in the four-space motion.

MILLS, MURA: For both of these resonant effects the periodicity of the resonance is zero. Then the driving force for the resonance is essentially an average of the forces along the axis. However, the average to be taken must be weighted by the Courant-Livingston-Snyder focusing parameters of the linear motion. Then one might expect a larger driving force in the ++-- quadrupole arrangement, where the ratio β_{max}/β_{ave} is large, than in the +-+- configuration that you calculated. This may tend to remove the apparent discrepancy between these results and the actual emittance blowup in the AGS Linac.

CHASMAN: I had not looked at that consideration at all.

LAPOSTOLLE: I would like to add something to what M. Promé said. The focusing he uses is ++--, and this focusing strength changes during acceleration.

Theta is not constant, generalizing in some way what has been said. Also, some estimates have been made by J. Faure showing the difference between a beam transport system and a linac. The fringing field effect is much more important in beam transport systems than it is in a linac due to the different relative dimensions of the emittance.

VAN STEENBERGEN, BNL: To restate a case, at the BNL linac the transverse emittance is measured routinely after the preinjector and after the linear accelerator. The average value of quite a number of measurements indicates a two-dimensional transverse phase space blowup equal to approximately 2.7. From what I have heard so far, no explanation of this exists as yet.

CHASMAN: I have actually done some computations using all the parameters for the 50-MeV linac, and what I have observed is in the order of a factor of 2, with an ellipse in the longitudinal phase space which corresponded to ±26°, which should be more or less the acceptance. I don't know how much the use of a buncher would change this, but I believe we should get some kind of estimate of the effect.

LAPOSTOLLE: In the CERN linac the growth in emittance is also of the order of a factor of three. We have not yet tried to compute or explain this increase numerically, but there are other things which may be responsible for it. Having again recently measured the field distribution and the exact position of the drift tubes in Tank One, it is clear that the phase motion is far from being very smooth. There are oscillations or irregularities of about ±10°, not so much due to misalignment as to the fact that at the time the design was made many facts were not clear concerning the exact motion. With this big phase oscillation, there are certainly other terms which can occur in the transverse motion. In addition, our quadrupoles are nonlinear.

CHASMAN: I have also examined CERN. I have tried to simulate the first tank. I observed a blowup of a factor of 3, like you said, starting with an ellipse in the longitudinal phase space corresponding to your synchronous phase of 30°.

GLUCKSTERN: I would like to add one more effect which could contribute to the growth in transverse phase space and that is due to the transverse misalignment. The main effect of transverse misalignments is to move the entire beam without change to some different transverse position. However, because of the fact that the equivalent transverse oscillations are larger, the coupling turns are larger in the analysis of the transverse-longitudinal coupling, and one gets additional growths which are some 20% of the motion due to misalignments.

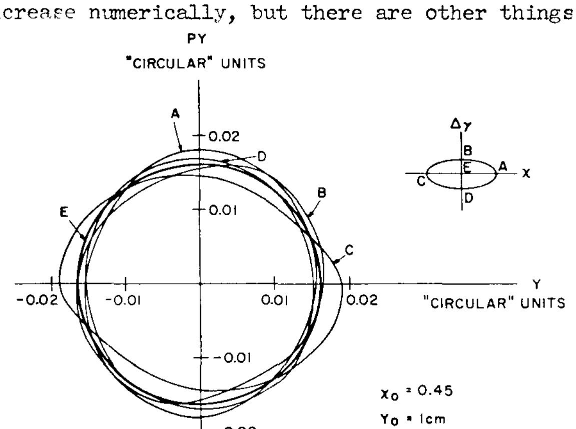

Fig. 1. Emittance in $y - p_y$ plane at 10 MeV for different initial longitudinal phases.

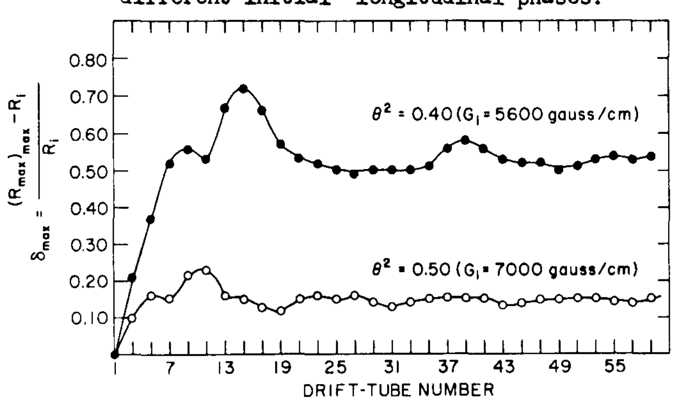

Fig. 2. Maximum fractional increase in transverse amplitude as function of drift tube number.

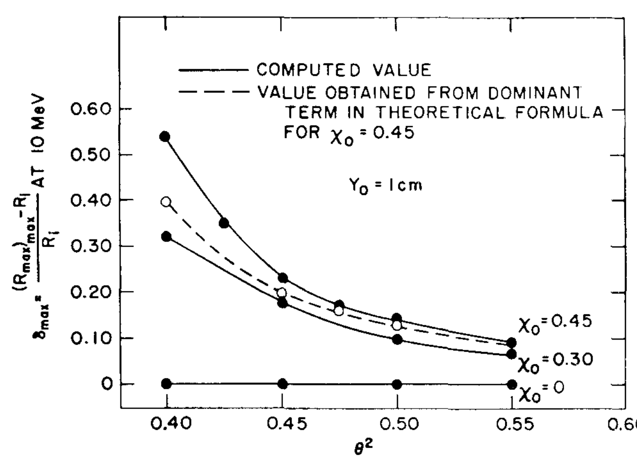

Fig. 3. Maximum fractional increase in transverse amplitude as function of θ^2.

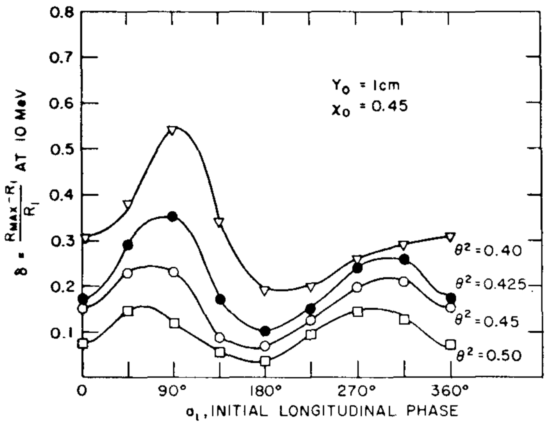

Fig. 4. Fractional increase in transverse amplitude as function of initial longitudinal phase.

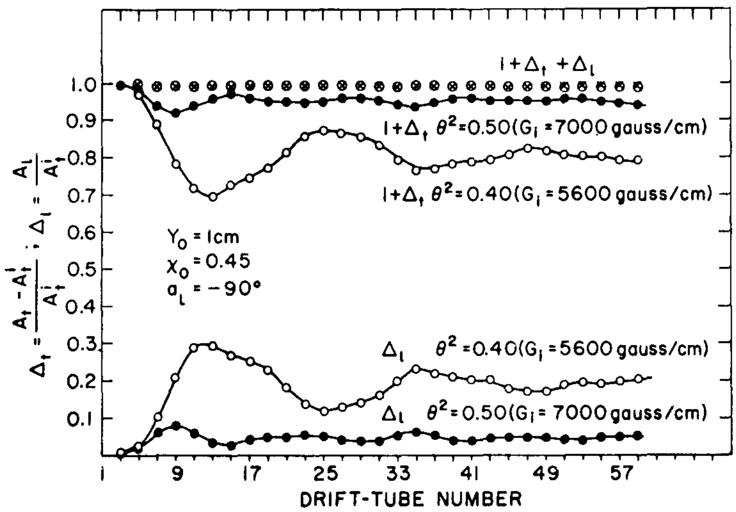

Fig. 5. Fractional change in actual phase space areas as function of drift tube number.

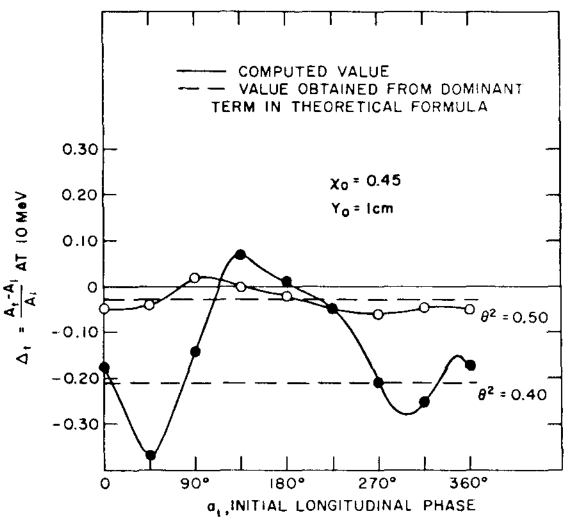

Fig. 6. Fractional change in transverse phase space as function of initial longitudinal phase.

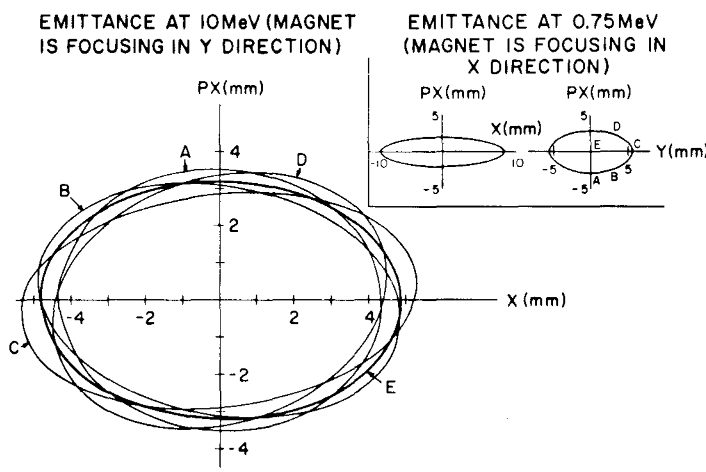

Fig. 7. Emittance in x – p_x plane at 10 MeV for different initial y – p_y coordinates.

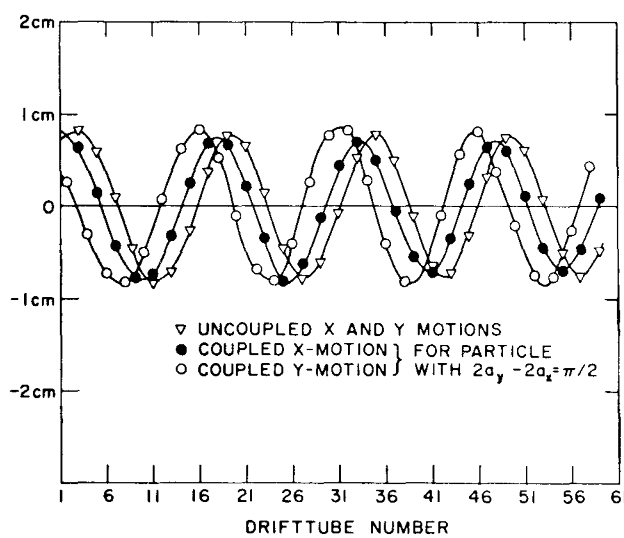

Fig. 8. Coupled and uncoupled x and y motions with $W_x = W_y$ as function of drift tube number.