COUPLED MOTION IN LOW-ENERGY PROTON LINEAR ACCELERATORS*

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I. Introduction

For most proton linear accelerators[1] currently operating or in the design stage, the injection velocity is in the range of 0.03c ~ 0.04c (500 ~ 750 keV) and the initial phase ψ of "stable" particles (that is, particles that will be accelerated up to the final energy) is between ~ -60° and ~ +30°. The transverse motion of each particle in the low energy region (say, below 10 MeV) is strongly influenced (sin ψ dependence of the rf defocusing force) by the large (longitudinal) phase oscillation which, in turn, is affected by the transverse motion through the dependence of the transit time factor on the transverse coordinate. In the Lagrangian which describes the motion of the particle, these effects are contained in coupling terms of the form[2]

$$r^{2n}\ (\psi-\psi_s)^m\ ,\quad n,\ m = 1,2,3,\ \ldots \qquad (1)$$

where r is the transverse coordinate of the particle and ψ_s is the synchronous phase around which the particle phase ψ oscillates.

Recently, R. L. Gluckstern has investigated analytically the coupling effect in proton linear accelerators[3-5] and, in conjunction with his work, a great deal of numerical calculations of the coupled particle motion has been performed by D. Swenson at Los Alamos and by R. Chasman at Brookhaven.[6] According to Gluckstern, the major effect of the coupling is an apparent increase in the (longitudinal and transverse) phase space area.[3] For example, if a beam is initially represented by an ellipse in the transverse phase space, it will become distorted and this distortion depends on the "history" of each particle's longitudinal phase oscillation. Consequently, the area of the transverse phase space remains constant (in the first approximation) for a single point in the longitudinal phase space ("fish") but will show an apparent increase for a group of different initial points in the "fish". Furthermore, this increase will be enhanced because of the misalignment errors in the focusing system.[4] If higher-order effects are taken into account, a single point in the "fish" (with zero area) will expand to a certain area and this in turn will introduce an essential change (either an increase or a decrease) in the transverse area.[5] The essential change of areas due to the "feed-back" effect is usually small[5,6] (a few percent) and does not seem to be of practical importance.

As has been emphasized by Gluckstern,[3] his analytical results have a number of inevitable limitations:

1. Linear approximations for uncoupled motions. This is particularly serious for longitudinal phase oscillations.**

2. Smooth approximation[7] of the strong focusing system. The uncoupled transverse motion is characterized by the maximum amplitude and the frequency k_t ($= 2\pi/\lambda_t$ where λ_t is the wavelength) only. The operating point in the transverse stability diagram[8,9] is limited to a small lateral region near the lower (small k_t) boundary. This is especially serious for (+) (+) (-) (-) arrangement of quadrupoles.[10]

3. Only the "asymptotic" value of the change in area is estimated and "local" behavior of coupling effects (before they become negligible) are hard to evaluate. An extensive numerical orbit computation which would utilize his analytical results as a guide has been suggested by Gluckstern for actual design works.

The purpose of this note is to study the same problem from a different viewpoint which is based on the theory of the strong focusing system by Courant and Snyder.[7] It is intended to be complementary to Gluckstern's work and to serve as a possible bridge between his analytical results and more elaborate numerical calculations. Only the apparent increase of the transverse phase space area is investigated, neglecting the "feed-back" effect. The uncoupled transverse motion is essentially linear but the uncoupled phase oscillation is solved exactly by numerical calculations. Magnet misalignments and the small coupling between two transverse motions (x and y) through the longitudinal motion are entirely neglected. Numerical examples given here are primarily for illustrating the method and should not be considered as a design of practical accelerators.

II. Periodic System

In describing the transverse motion, it is customary to take the coordinate x and its derivative $x' \equiv dx/dz$ as a function of z. However, when the particle is accelerated, it is more convenient to use x and $p_x \equiv px'$, p being the momentum of the particle in units of m_0c. Equations of the transverse motion are

$$(dx/dz) \equiv x' = p_x/p(z) \tag{2}$$

$$(dp_x/dz) \equiv p_x' = f(z,r)x + q(z)x \tag{3}$$

where $f(z,r)$ and $q(z)$ represent the rf field and quadrupole magnets, respectively. Since the "feed-back" effect is neglected, quantities for the longitudinal motion (velocity, phase, etc.) which affect $f(z,r)$ and $q(z)$ are all independent of (x,p_x). Furthermore, one can usually neglect higher-order terms (in r) of $f(z,r)$,

$$f(z,r) \cong f(z,r=0). \tag{4}$$

Combining Eqs. (2) and (3), one gets

$$px'' + p'x' = F(z)x \tag{5}$$

where

$$F(z) = f(z) + q(z). \tag{6}$$

For a strictly periodic system with the period L, $F(z+L) = F(z)$ and $p(z+L) = p(z)$. One can easily follow the formalism of CS (Sections 2 and 3) to get the general solution of (5) in the form

$$x(z) = \sqrt{W\beta_p(z)}\cos[\varphi(z)+\delta], \tag{7}$$

$$p_x(z) = -\left[\sqrt{W/\beta_p(z)}\right]\left\{\sin[\varphi(z)+\delta] + \alpha_p\cos[\varphi(z)+\delta]\right\}, \tag{8}$$

with arbitrary constants W and δ. In matrix notation,

$$Y(z_1) \equiv \begin{bmatrix} x(z_1) \\ p_x(z_1) \end{bmatrix} = M_x(z_1/z)Y(z), \tag{9}$$

one is particularly interested in the matrix $M_x(z) \equiv M_x(z+L/z)$ which describes the particle motion through a whole period,

$$Y(z+L) = M_x(z)Y(z), \tag{10}$$

$$M_x(z) = \begin{bmatrix} \cos\mu + \alpha_p\sin\mu, & \beta_p\sin\mu \\ -\gamma_p\sin\mu, & \cos\mu - \alpha_p\sin\mu \end{bmatrix} \tag{11}$$

where $\alpha_p \equiv \alpha_p(z)$ etc. and the (transverse) phase advance μ is related to the frequency k_t by

$$\mu = \int_z^{z+L} dz/p\beta_p = k_tL. \tag{12}$$

Betatron oscillation parameters α_p, β_p and γ_p are slightly different from corresponding parameters in CS:

$$\beta_p\gamma_p - \alpha_p^2 = 1 \tag{13}$$

$$\beta_p' = -2\alpha_p/p, \tag{14}$$

$$\alpha_p' = -\beta_pF - \gamma_p/p = -\beta_pF - (1+\alpha_p^2)/p\beta_p, \tag{15}$$

$$\gamma_p' = -2F\alpha_p, \tag{16}$$

$$\varphi(z) = \int_0^z dz/p\beta_p. \tag{17}$$

$$\mu = \varphi(z=L) \tag{18}$$

It is clear from Eqs. (7) and (8) that the quantity

$$W = \gamma_p(z)x^2 + 2\alpha_p(z)xp_x + \beta_p(z)p_x^2 \tag{19}$$

is a constant of the motion. If a beam occupies the elliptic shape given by (19) in (x,p_x) phase space at z, it occupies a different ellipse (corresponding to different values of α_p, β_p, and γ_p) at other points but the area πW remains constant. After one period, at z + L, the shape is back to the ellipse (19).

III. Adiabatic Invariant for Approximately Periodic System

In the actual linear accelerator, especially at low energies, the momentum p and the function F(z) are not strictly periodic. The particle is continuously accelerated and the parameters of the focusing system vary from one period to the next. In the stability diagram of Smith and Gluckstern,[8] the operating point oscillates laterally with the longitudinal phase oscillation. Also, it may move vertically, as the particle is accelerated, in accordance with a particular choice of magnet parameters through the accelerator.[9] The quantity W, Eq. (19), is no longer a constant of the motion, its value changing from period to period. However, when the deviation from the strict periodicity is small, that is, when magnet parameters change slowly and the frequency k_ℓ of the longitudinal phase oscillation is small compared to the transverse frequency k_t, it is still possible to find an adiabatic invariant, at least to the lowest order of the deviation. The derivation of this adiabatic invariant is given in Section 3(d) of CS. (See also Ref. 11.)

The quantity W defined by Eq. (19) can still be regarded as constant in each period. The change of this quantity from k-th period to (k + 1)th period is, to lowest order of the variation,

$$W_{k+1} = W_k\left\{1 - \frac{(\Delta\beta)_k}{\beta_k}\cos(2\mu_k+2\delta_k) - \left[(\Delta\alpha) - \frac{(\Delta\beta)}{\beta}\alpha\right]_k \sin(2\mu_k+2\delta_k)\right\}, \quad (20)$$

where $(\Delta\beta)_k = \beta_{k+1} - \beta_k$, etc. and the subscript p of the betatron oscillation parameters is left out. The constant phase δ_k is defined such that, at the beginning of the k-th period, x and p_x are given by Eqs. (7) and (8), respectively, with $\varphi(z) = 0$. It is approximately equal to the total phase advance up to the beginning of the k-th period plus the initial phase δ_1,

$$\delta_k = [\sum_{i=1}^{k-1} \mu_i] + \delta_1 + \mathcal{O}(\Delta\beta/\beta,\ \Delta\alpha/\alpha) \quad (21)$$

so that

$$\mu_k + \delta_k = [\sum_{i=1}^{k} \mu_i] + \delta_1 \equiv \varphi_k + \delta_1 . \quad (22)$$

When the operating point is near the optimum region ($\mu \simeq \pi/2$) for (+) (-) (+) (-) system, μ_i is practically constant and

$$\varphi_k \equiv \sum_{i=1}^{k} \mu_i \simeq k\mu \quad (23)$$

For a cavity with (N + 1) periods, the final value of W is related to its initial value by

$$W_{N+1} = W_1\left\{1 - \varepsilon_N \cos(2\delta_1 + a_N)\right\}, \quad (24)$$

where

$$\varepsilon_N = (C_N^2 + S_N^2)^{1/2}, \quad a_N = \tan^{-1}(S_N/C_N), \quad (25)$$

$$C_N = \sum_1^N \frac{(\Delta\beta)_k}{\beta_k}\cos(2\varphi_k) - \sum_1^N \alpha_k\left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)_k \sin(2\varphi_k), \quad (26)$$

$$S_N = \sum_1^N \frac{(\Delta\beta)_k}{\beta_k}\sin(2\varphi_k) + \sum_1^N \alpha_k\left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)_k \cos(2\varphi_k). \quad (27)$$

At the end of the (N + 1)th period,

$$x = \sqrt{W_{N+1}\ \beta_{N+1}}\cos\theta_{N+1} \quad (28)$$

$$p_x = -\left(\sqrt{W_{N+1}/\beta_{N+1}}\right)(\sin\theta_{N+1} + \alpha_{N+1}\cos\theta_{N+1}) \quad (29)$$

with

$$\theta_{N+1} = \varphi_{N+1} + \delta_1 + \mathcal{O}(\Delta\beta/\beta,\ \Delta\alpha). \quad (30)$$

Equation (24) is similar to Eq. (15) of Ref. 4 when misalignment effects (D_i) are ignored.

IV. Effective Increase of Phase Space Area

In the preceding section, the variation of the quantity W has been considered only for a single particle. Given initial conditions $x(z = 0)$ and $p_x(z = 0)$, or, equivalently [see Eqs. (7) and (8)], W_1 and δ_1, one can calculate the maximum possible amplitude $|x|_{max}$ in the (N + 1)th period from Eqs. (24) and (28):

$$|x|_{max} = \sqrt{W_{N+1}\quad \text{max. } \beta_{N+1}} \quad (31)$$

where max. β_{N+1} is the maximum value of the parameter $\beta(z)$ in the (N + 1)th period. It is assumed here that values of μ_k, α_k, β_k, and γ_k are already known for the entire cavity, $k = 1, 2, \ldots, N + 1$.[10,12] However, in the design of focusing systems for proton linear accelerators, one is more interested in the transverse phase space area occupied by a beam, that is, a collection of many particles with different longitudinal phase values.

The initial conditions of all particles of the beam with a given area πW_s in (x, p_x) phase space can be taken such that

$$\gamma_s x^2 + 2\alpha_s x p_x + \beta_s p_x^2 \leq W_s \quad (32)$$

where γ_s, α_s and β_s are betatron oscillation parameters in the first period [see Eqs. (13) - (17)] for the synchronous particle. In terms of W_s and the phase δ_s, each point on the boundary can be expressed in the form

$$x = \sqrt{W_s\quad \beta_s}\cos\delta_s \quad (33)$$

$$p_x = -\left(\sqrt{W_s/\beta_s}\right)(\sin\delta_s + \alpha_s\cos\delta_s) \quad (34)$$

with $0 \leq \delta_s \leq 2\pi$. On the other hand, the same point (x, p_x) can be written in the form of Eqs. (7) and (8) using W_1 and the initial phase δ_1

$$x = \sqrt{W_1\quad \beta_1}\cos\delta_1 \quad (35)$$

$$p_x = -\left(\sqrt{W_1/\beta_1}\right)(\sin\delta_1 + \alpha_1\cos\delta_1) \quad (36)$$

where α_1 and β_1 are betatron oscillation parameters in the first period corresponding to a particular point in the longitudinal phase space. Particles that are distributed on the boundary of the ellipse [Eq. (32) with equal sign] but correspond to a single non-synchronous point in the longitudinal phase space have in general different δ_1 and W_1. Also, a single point on the ellipse represents many particles in the longitudinal phase space and each particle has a different set of (W_1, δ_1). From Eqs. (33) - (36), one can show that

$$W_1/W_s = 1 - \varepsilon \cos (2\delta_s + a) \qquad (37)$$

where

$$\varepsilon = \sqrt{(\delta\beta/\beta_s)^2 + \alpha_s^2\left[(\delta\beta/\beta_s) - (\delta\alpha/\alpha_s)\right]^2} \qquad (38)$$

$$a = \tan^{-1}\left[\alpha_s \left(\frac{\delta\beta}{\beta_s} - \frac{\delta\alpha}{\alpha_s}\right)/(\delta\beta/\beta_s)\right] \qquad (39)$$

$$(\delta\beta) \equiv \beta_1 - \beta_s \ , \quad (\delta\alpha) \equiv \alpha_1 - \alpha_s \qquad (40)$$

and

$$\delta_1 \equiv \delta_s + \Delta = \delta_s + \left\{\left(\frac{\delta\beta}{\beta_s}\right) \sin \delta_s \cos \delta_s - (\delta\alpha) \cos^2 \delta_s + \alpha_s \frac{(\delta\beta)}{\beta_s} \cos^2 \delta_s\right\} . \qquad (41)$$

Combining Eq. (37) with Eq. (24), one gets, at the end of the (N + 1)th period,

$$W_{N+1} \simeq W_s \left\{1 - \bar{\varepsilon} \cos (2\delta_s + \bar{a})\right\} \qquad (42)$$

with

$$\bar{\varepsilon} = \sqrt{\varepsilon^2 + \varepsilon_N^2 + 2\varepsilon\varepsilon_N \cos (a - a_N)} \qquad (43)$$

$$\bar{a} = \tan^{-1}\left[(\varepsilon \sin a + \varepsilon_N \sin a_N)/(\varepsilon \cos a + \varepsilon_N \cos a_N)\right] . \qquad (44)$$

It should be noted here that, since the "feed-back" effect is neglected, all particles represented by a single point in the longitudinal phase space have the same values of ε, ε_N, a, and a_N. Since they are initially distributed on the boundary of the ellipse (32) in the transverse phase space, the phase δ_s varies from 0 to 2π and the value of the quantity W_{N+1} ranges from $W_s\ (1 - \bar{\varepsilon})$ to $W_s\ (1 + \bar{\varepsilon})$. Particles that are inside of the ellipse simply correspond to a smaller value of W_s. From Eqs. (28), (29), and (42), it can be shown that, at the end of (N + 1)th period, these particles are on a distorted ellipse (in the first-order approximation)

$$A\ x^2 + 2Hxp_x + B\ p_x^2 = W_s \qquad (45)$$

where

$$A = \gamma_{N+1} + \bar{\varepsilon}\left\{(1 - \alpha_{N+1}^2) \cos \theta + 2\alpha_{N+1} \sin \theta\right\}/\beta_{N+1} \ , \qquad (46)$$

$$H = \alpha_{N+1} + \bar{\varepsilon}\ (\sin \theta - \alpha_{N+1} \cos \theta) \ , \qquad (47)$$

$$B = \beta_{N+1}\ (1 - \bar{\varepsilon} \cos \theta) \ , \qquad (48)$$

and

$$\theta = \bar{a} - 2\varphi_{N+1} + \mathcal{O}(\Delta\beta/\beta,\ \Delta\alpha\ ;\ \delta\beta/\beta_s,\ \delta\alpha) . \qquad (49)$$

To the lowest order in $\bar{\varepsilon}$, the area of ellipse (45) is the same (πW_s) as the initial value [Eq. (32)]. Since the "feed-back" is neglected, the area should of course be strictly conserved. On the other hand, the ellipse is distorted and the maximum values of x and p_x are affected by the coupling effect:

$$|x|_{max} = \sqrt{W_s\ \ B} = \sqrt{W_s\ \ \beta_{N+1}}\sqrt{1 - \bar{\varepsilon} \cos \theta} \qquad (50)$$

$$|p_x|_{max} = \sqrt{W_s\ \ A} = \sqrt{W_s\ \ \gamma_{N+1}} \times \sqrt{1 + \bar{\varepsilon}\left\{(1-\alpha^2)\cos \theta + 2\alpha \sin \theta\right\}/(1+\alpha^2)} . \qquad (51)$$

If one now considers certain distribution of particles in the longitudinal phase space also, one would get ellipses of the form (45) with different values of A, B, and H corresponding to different values of ε, ε_N, a, a_N, and φ_{N+1}. If phase oscillations are still important at this point, values of α, β, and γ would also be different. The original ellipse (32) is then transformed to a collection of ellipses (45), each having the same area πW_s but with different orientations and deformations. The net result is an effective increase in the transverse phase space area by approximately a factor of $(1 + \bar{\varepsilon}_{max})$. When phase oscillations are negligible at the end of (N + 1)th period, parameters α_{N+1}, β_{N+1}, and γ_{N+1} are the same for all particles and the distribution of particles in the transverse phase space is

$$X^2 + P_x^2 \simeq W_s\,(1 + \bar{\varepsilon}_{max}) \qquad (52)$$

where

$$X = x/\sqrt{\beta_{N+1}} \qquad (53)$$

and

$$P_x = (\alpha_{N+1}\, x + \beta_{N+1}\, p_x)/\sqrt{\beta_{N+1}}\ . \qquad (54)$$

V. Application

In order to illustrate a possible use of the formalism given in preceding sections, a somewhat artificial linac cavity has been designed which is based on the MURA linac cavity calculations. The main characteristics of the cavity are:

Frequency of the rf = 201.25 MHz
Injection velocity = 0.04c (0.75 MeV)
Final velocity = 0.144c (9.90 MeV)
Total number of unit cells = 56
Cavity diameter = 0.94 m
Drift tube diameter = 0.18 m
Bore diameter = 2.0 cm
Total length = 7.18 m
Average axial field $E_o(n)$ = 1.40 + 0.012 (n-1) MeV
(where n is the cell number)
Transit time factor = 0.584 ~ 0.811 .

The focusing system is (+) (-) (+) (-) with quadrupoles in all drift tubes (including half drift tubes at both ends) and their strength is given by

$$H'\ (\mathrm{kG/cm}) = 0.282\ (\gamma/\beta)\ , \qquad (55)$$

where γ = total energy/m_oc^2 and β = velocity/c at each magnet. The length of each magnet is taken to be one-half of the cell length.

In each cell, the electric field on the axis, $E_z(z,r=0)$, has been written in the form

$$E_z(z,\ r=0) = E_o(n)\Big\{1 + \sum_{m=1}^{4} e_m(n)\cos\left(\frac{2\pi m}{L_n}\, z\right)\Big\}\ . \qquad (56)$$

with

$$e_m(n) = \sum_{k=0}^{2} a_{mk}\, n^k \qquad (57)$$

and coefficients $\{a_{mk};\ m = 1\text{-}4,\ k = 0\text{-}2\}$ have been obtained from numerical values of $E_z(z,\ r=0)$ for n = 1, 27, and 56. These values are results from MESSYMESH calculations at MURA. For each value of n, two cells whose frequencies are close to 201.25 MHz have been used to get linearly interpolated values of e_m. Below 10 MeV, higher harmonic components in Eq. (56) [$e_m(n)$ for $m \geq 5$] are very small and they can be ignored for beam dynamics calculations. The drift tube table has been prepared by an exact numerical integration of the phase motion[2] such that a particle with the velocity = 0.04c and the phase φ_o at the injection keeps the same phase φ_o (mod. 2π) at the beginning of each cell. With $\varphi_o = -\pi - 0.48$, the phase of the particle at the center of each cell varies from -0.44 in the first cell to -0.46 in the last cell. This particle will be called the "synchronous" particle. Once $E_z(z,\ r=0)$ is fixed [Eq. (56)], it is easy to obtain $E_z(z,\ r\neq0)$, $E_r(z,r)$, and $H_\theta(z,r)$ from Maxwell's equations. Betatron oscillation parameters α_p, β_p, γ_p, and μ_p for each period (two cells) have been calculated from Eqs. (10) and (11) where two independent sets of solutions $Y_1(z = L)$ and $Y_2(z = L)$ can be obtained by numerically integrating longitudinal and transverse equations of motion. Since the "feed-back" effect is neglected in the formalism, the longitudinal equations of motion are independent of the transverse motion.

Three points in the longitudinal phase space have been studied in addition to the synchronous particle. Their initial positions are:

Synchronous Particle	$\Delta\gamma=0$ (velocity = 0.04c), $\varphi=\varphi_o=-\pi-0.48$
Particle 1	$\Delta\gamma=0$, $\Delta\varphi=\varphi-\varphi_o$ = -0.408
Particle 2	$\Delta\gamma=0$, $\Delta\varphi$ = 0.446
Particle 3	$\Delta\gamma=0$, $\Delta\varphi$ = 0.892

Particles 1 and 3 are very close to boundaries of the stable area. The parameter β_p along the entire cavity for these particles is shown in Fig. 1 where one can clearly see the variation of β_p due to the longitudinal phase oscillation. The quantity $\bar{\varepsilon}$ defined by Eq. (43) is given in Table I for Particles 1-3 and the effective increase in the transverse phase space area at the end of the cavity is 24%. For the synchronous particle, ε = 0 [Eq. (38)] and $\varepsilon_N \simeq 0$ since $\Delta\beta/\beta$ and $\Delta\alpha/\alpha$ are very small for all values of k. [Eqs. (25)-(27)]. Therefore $\bar{\varepsilon} \simeq 0$. Finally,

Table I***

	Particle 1	Particle 2	Particle 3
β_1	7.250 m	5.477 m	4.898 m
α_1	-0.02673	-0.03427	-0.02921
ε	0.1502	0.1316	0.2233
a	-0.07355	-0.04369	-0.01655
ε_N	0.07641	0.00694	0.02603
a_N	-0.7270	1.335	-1.1413

$\bar{\varepsilon}$	0.2159	0.1331	0.2357

$(\beta_s)_1 = 6.306$ m, $(\alpha_s)_1 = -0.03284$

N = 27

values of the quantity ε_k, k = 1 - N (= 27), for Particles 1 and 3 are shown in Fig. 2. Since $\bar{\varepsilon} \simeq \varepsilon + \varepsilon_k$ [Eq. (43)] and ε is given in Table I, one can calculate the effective increase in the transverse phase space area at each period.

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References

1. Lloyd Smith, Handbuch der Physik, Volume XLIV, pp. 341-389.
2. S. Ohnuma, Internal Report Y-15, Yale University (to be issued).
3. R. L. Gluckstern, Internal Report AADD-31, Brookhaven National Laboratory, April 2, 1964. This has been issued by Yale as Y-11.
4. R. L. Gluckstern, Internal Report MP-4/RLG-5, Los Alamos Scientific Laboratory, March 14, 1966.
5. R. L. Gluckstern, Internal Report (to be issued). Brookhaven National Laboratory, also private communication.
6. D. Swenson and R. Chasman, private communication. Part of their results were reported at the informal meeting on proton linacs at Los Alamos in March, 1966. The author is grateful to them for showing me their results and for their discussions on related subjects.
7. E. D. Courant and H. S. Snyder, Annals of Physics <u>3</u>, 1 (1958). This paper will be referred to as CS in this note.
8. L. Smith and R. L. Gluckstern, Rev. Sci. Inst. <u>26</u>, 220 (1955).
9. L. Smith, <u>loc</u>. <u>cit</u>., pp. 376-377.
10. S. Ohnuma, Internal Report Y-13, Yale University, May 20, 1966.
11. S. Ohnuma, Internal Report Y-14, Yale University, June 30, 1966.
12. J. N. Vitale, Technical Note JNV-4, Yale University, July 23, 1964.

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** It has recently been found by Gluckstern that this is indeed the most serious defect in his calculation. More accurate results were reported by him at the Los Alamos Conference.

*** Results given here and in Fig. 2 are for particles with δ_1 = 0 or $\pi/2$.

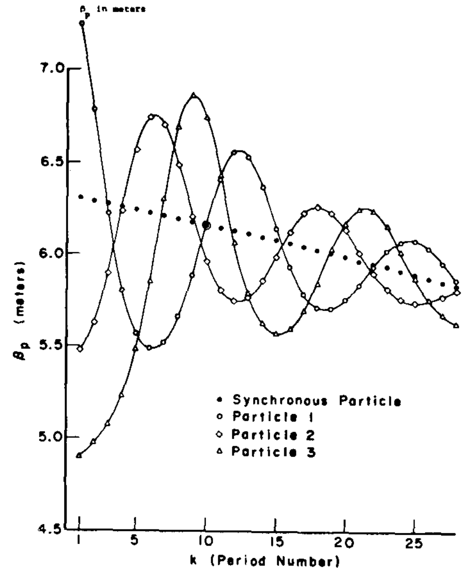

Fig. 1. Betatron oscillation parameter β_p (Eq. 11) for four particles in low energy (0.75 ~ 10 MeV) cavity. Period number k is twice cell number since focusing magnet configuration is (+) (-) (+) (-). Initial positions are:

Synchr. Particle $\Delta\gamma$=0 (0.75 MeV), $\varphi = -\pi-0.48 \equiv \varphi_0$

Particle 1: $\Delta\gamma$=0, $\Delta\varphi \equiv \varphi-\varphi_0 = -0.408$

Particle 2: $\Delta\gamma$=0, $\Delta\varphi$ = 0.446

Particle 3: $\Delta\gamma$=0, $\Delta\varphi$ = 0.892

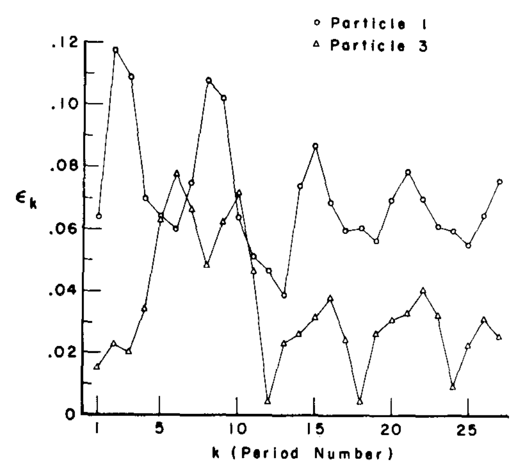

Fig. 2. Fractional change ϵ_k (Eqs. 24 and 25) of quantity $W = \gamma x^2 + 2\alpha x p_x + \beta p_x^2$ (Eq. 19) as function of period number. Cavity and particles are same as in Fig. 1. Effective increase in transverse phase space area at each period k is $\bar{\varepsilon} \simeq \varepsilon + \varepsilon_k$ where ε = 0.150 and 0.223 for Particles 1 and 3, respectively. Initial transverse phase angle δ_1 is 0 or $\pi/2$.