CUMULATIVE BEAM BREAKUP WITH TIME-DEPENDENT PARAMETERS*

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Abstract

A general analytical formalism developed recently for cumulative beam breakup (BBU) in linear accelerators with arbitrary beam current profile and misalignments [1, 2] is extended to include time-dependent parameters, such as energy chirp or rf focusing, in order to reduce BBU-induced instabilities and emittance growth. Analytical results are presented and applied to practical accelerator configurations.

FORMULATION AND SOLUTION

In a continuum approximation, the transverse motion of a relativistic beam under the influence of focusing and BBU can be modeled by [1]

$$\left[\frac{1}{\gamma}\frac{\partial}{\partial\sigma}\left(\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2\right]x(\sigma,\zeta) = \varepsilon\int_0^\zeta d\zeta_1\, w(\zeta-\zeta_1)F(\zeta_1)\,x(\sigma,\zeta_1) \tag{1}$$

where γ is the usual energy parameter; $\sigma = s/\mathbf{L}$, is the distance from the front of the accelerator normalized to the accelerator length; κ is the normalized focusing wave number; $\zeta = \omega\left(t-\int ds/\beta c\right)$, is the time made dimensionless by the frequency ω and measured after the arrival of the head of the beam at location σ; $F(\zeta) = I(\zeta)/\bar{I}$, the current form factor, is the instantaneous current divided by the average current; $w(\zeta)$ is the wake function, which, in the case of a single dipole mode, is assumed to be $w(\zeta) = u(\zeta)\,\sin\zeta\; e^{-\zeta/2Q}$; ε is the coupling strength between the beam and the dipole mode, and includes properties of the beam and the deflecting mode of the accelerating structure.

While Eq. (1) assumes a perfectly aligned accelerator, misalignment of the cavities and focusing elements can also be included in the following analysis in a straightforward fashion.

Without loss of generality, we will assume a coasting beam. As shown in [1], the analytical results can be extended to an accelerated beam by suitable coordinate and variable transformations. Under these assumptions, the equation of motion becomes

$$\frac{\partial^2}{\partial\sigma^2}x(\sigma,\zeta)+\kappa^2 x(\sigma,\zeta) = \varepsilon\int_0^\zeta d\zeta_1\, w(\zeta-\zeta_1)\,F(\zeta_1)\,x(\sigma,\zeta_1). \tag{2}$$

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In the following we will make the additional assumption that the injection offsets (lateral displacement and angular divergence) are time-independent. Again, time-dependent injection parameters can be included in the following formalism.

Time-independent Parameters

In [1], Equation (2) was solved under the assumption of constant, time-independent BBU coupling and focusing strengths (ε and κ). Applying to Eq. (2) the Laplace transform with respect to $\sigma : \mathcal{L}\left[x(\sigma,\zeta)\right] = x^\dagger(p,\zeta)$, we obtain

$$x^\dagger(p,\zeta) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{\left(p^2+\kappa^2\right)^{n+1}}\left[\,x_0\,p + x_0'\,\right] f_n(\zeta), \tag{3}$$

with

$$f_{n+1}(\zeta) = \int_0^\zeta f_n(\zeta_1)\,w(\zeta-\zeta_1)\,F(\zeta_1)\,d\zeta_1, \qquad f_0(\zeta) = 1. \tag{4}$$

Applying the inverse Laplace transform gives

$$x(\sigma,\zeta) = \sum_{n=0}^{\infty}\varepsilon^n\left[x_0 j_n(\kappa,\sigma) + x_0' i_n(\kappa,\sigma)\right] f_n(\zeta) \quad , \tag{5}$$

where

$$i_n(\kappa,\sigma) = \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \frac{1}{\kappa}\sqrt{\frac{\pi\kappa\sigma}{2}}\,J_{n+\frac{1}{2}}(\kappa\sigma) \;, \qquad j_n(\kappa,\sigma) = \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}}\,J_{n-\frac{1}{2}}(\kappa\sigma) \;. \tag{6}$$

Time-dependent Parameters

When the BBU coupling and focusing strengths are time-dependent [$\varepsilon(\zeta)$ and $\kappa(\zeta)$] the beam displacement $x(\sigma,\zeta)$ is not given by Eqs. (3)-(6) anymore and the procedure for solving Eq. (2) needs to be modified. This can be done simply by splitting the focusing strength $\kappa(\zeta)$ in two parts, one constant and the other time-dependent, such that:

$$\kappa^2(\zeta) = \kappa_0^2\left[1+\Delta\kappa(\zeta)\right] \tag{7}$$

The displacement $x(\sigma,\zeta)$ and its Laplace transform $x^\dagger(p,\zeta)$ are then given by

$$x^\dagger(p,\zeta) = \sum_{n=0}^{\infty}\frac{\left[\,x_0\,p + x_0'\,\right]}{\left(p^2+\kappa_0^2\right)^{n+1}} f_n^*(\zeta) \quad , \tag{8}$$

$$x(\sigma,\zeta) = \sum_{n=0}^{\infty}\left[x_0 j_n(\kappa_0,\sigma) + x_0' i_n(\kappa_0,\sigma)\right] f_n^*(\zeta), \tag{9}$$

$$f_{n+1}^{*}(\zeta)=\varepsilon(\zeta)\int_0^{\zeta} f_n^{*}(\zeta_1)w(\zeta-\zeta_1)\,F(\zeta_1)\,d\zeta_1 -\kappa_0^2\,\Delta\kappa(\zeta)\,f_n^{*}(\zeta), \tag{10}$$

$$f_0^{*}(\zeta)=1,$$

$$i_n(\kappa_0,\sigma)=\frac{1}{n!}\left(\frac{\sigma}{2\kappa_0}\right)^n\frac{1}{\kappa_0}\sqrt{\frac{\pi\kappa_0\sigma}{2}}J_{n+\frac{1}{2}}(\kappa_0\sigma)\ ,$$
$$j_n(\kappa_0,\sigma)=\frac{1}{n!}\left(\frac{\sigma}{2\kappa_0}\right)^n\sqrt{\frac{\pi\kappa_0\sigma}{2}}J_{n-\frac{1}{2}}(\kappa_0\sigma)\ . \tag{11}$$

There is some arbitrariness in the way the focusing strength $\kappa(\zeta)$ is split in two parts according to Eq. (7). For example, it could be assumed that $\kappa(\zeta)$ has no constant term $(\kappa_0=0)$ and only a time-dependent part. In this case $x(\sigma,\zeta)$ would be given by

$$x(\sigma,\zeta)=\sum_{n=0}^{\infty}\left[x_0\,j_n(0,\sigma)+x_0'\,i_n(0,\sigma)\ \right]f_n^{\circ}(\zeta), \tag{12}$$

with

$$f_{n+1}^{\circ}(\zeta)=\varepsilon(\zeta)\int_0^{\zeta} f_n^{\circ}(\zeta_1)w(\zeta-\zeta_1)\,F(\zeta_1)\,d\zeta_1 -\kappa^2(\zeta)\,f_n^{\circ}(\zeta), \tag{13}$$

$$f_0^{\circ}(\zeta)=1\,,$$

$$i_n(0,\sigma)=\frac{\sigma^{2n+1}}{(2n+1)!},\qquad j_n(0,\sigma)=\frac{\sigma^{2n}}{(2n)!}. \tag{14}$$

While expressions (9)-(11) and (12)-(14) for $x(\sigma,\zeta)$ look quite different they are mathematically equivalent and represent the same solution of Eq. (2). They differ however in the speed of convergence with (12)-(14) converging very slowly. For expressions (9)-(11) to be of practical use the separation of $\kappa(\zeta)$ in two parts, as given by Eq. (7), needs to be done in such a way that the time-dependent part $\Delta\kappa(\zeta)$ is kept as small as possible.

Form Eq. (10) we see that

$$f_1^{*}(\zeta)=\varepsilon(\zeta)\int_0^{\zeta} w(\zeta-\zeta_1)\,F(\zeta_1)\,d\zeta_1-\kappa_0^2\,\Delta\kappa(\zeta), \tag{15}$$

and choosing a time-dependent focusing such that

$$\kappa_0^2\,\Delta\kappa(\zeta)=\varepsilon(\zeta)\int_0^{\zeta} w(\zeta-\zeta_1)\,F(\zeta_1)\,d\zeta_1 \tag{16}$$

will yield $f_{n>0}^{*}(\zeta)=0$ and

$$x(\sigma,\zeta)=x_0\cos\kappa_0\sigma+x_0'\frac{\sin\kappa_0\sigma}{\kappa_0}\quad . \tag{17}$$

Equation (16) is the general condition for eliminating cumulative BBU by BNS damping [3].

SINGLE SHORT BUNCH

In the case of a single very short bunch, the wakefield can be assumed to be linear $\left[w(\zeta)=\zeta\right]$. If one assumes further that the bunch charge density is constant $[F(\zeta)=1]$, that the BBU coupling strength ε is constant and that the time-dependent focusing is of the form

$$\kappa^2(\zeta)=\kappa_0^2\left[1+\eta\,\zeta^2\right], \tag{18}$$

then the functions $f_n^{*}(\zeta)$ can be easily calculated:

$$f_n^{*}(\zeta)=\frac{\zeta^{2n}}{(2n)!}\prod_{k=1}^{n}\left[\varepsilon-(2k-1)(2k)\,\kappa_0^2\,\eta\right] \tag{19}$$

This, together with Eqs. (9) and (11), defines completely the displacement $x(\sigma,\zeta)$. If η is chosen such that $\eta=\varepsilon/(2\kappa_0^2)$, then $f_{n>0}^{*}(\zeta)=0$ and the coupling between the beam and the dipole mode is suppressed.

In the case of a linear time dependence of the focusing

$$\kappa^2(\zeta)=\kappa_0^2\left[1+\eta\zeta\right], \tag{20}$$

the functions $f_n^{*}(\zeta)$ can be obtained through the recurrence relations

$$f_n^{*}(\zeta)=\sum_{k=n}^{2n}a_{n,k}\zeta^k\ ,$$
$$a_{0,0}=1\ , \tag{21}$$
$$a_{n,k}=\frac{\varepsilon a_{n-1,k-2}}{k(k-1)}-\kappa_0^2\eta a_{n-1,k-1}\ .$$

FINITE TRAIN OF POINT-LIKE BUNCHES

The results of the previous sections will be applied here to a finite train of N identical point-like bunches separated, in the laboratory frame, by τ, so that bunch M is defined by $\zeta=M\omega\tau$. The displacement of bunch M is then given by

$$x_M(\sigma)=\sum_{n=0}^{\infty}f_n^{*}(M\omega\tau)\left[x_0\,j_n(\kappa_0,\sigma)+x_0'\,i_n(\kappa_0,\sigma)\ \right] \tag{22}$$

$$f_{n+1}^{*}(M\omega\tau)=\omega\tau\,\varepsilon(M\omega\tau)\sum_{k=0}^{M}f_n^{*}(k\omega\tau)\,w[(M-k)\omega\tau] -\kappa_0^2\,\Delta\kappa(M\omega\tau)f_n^{*}(M\omega\tau) \tag{23}$$

Table 1: Nominal top-level linear-collider design parameters [1,4,5]

Parameters	Value
Total initial energy	10 GeV
Total final energy	1 TeV
Linac length **L**	10 km
Number of betatron periods	100
Bunch charge	1 nC
Number of bunches N	90
Bunch spacing τ	2.8 ns
Deflecting-wake frequency $\omega/2\pi$	14.95 GHz
Deflecting-wake quality factor Q	∞
Deflecting-wake amplitude w_0	10^{15} VC^{-1}m^{-2}

As an example, the analytical results expressed by Eqs. (22) and (23) will be applied to a beam representative of a linear collider. For comparison, we will use the same parameters as those used in [1, 4, 5], and which are listed in Table 1. The time-dependent focusing is assumed to be of the form

$$\kappa^2(M\omega\tau) = \kappa_0^2\left[1+\eta\frac{M}{N-1}\right] \quad (24)$$

where η represents the relative variation of the focusing strength during the bunch train.

Results of the application of Eqs. (22) and (23) to the beam described in Table 1 are shown in Fig. 1 for $\eta = 0$, 0.01, 0.02, and 0.03. Since this is an accelerated beam, the variable and coordinate transformations described in Appendix A of [1] were applied to Eq. (22); in particular Fig. 1 shows the effect of adiabatic damping. The lower plot of Fig. 1 $(\eta = .03)$, which was obtained by direct calculation using the analytical results given by Eqs. (22) and (23), is identical to Fig. 4 of [5] which was obtained numerically by tracking successive bunches as they progress along the accelerator.

The incorporation of a finite Q for the deflecting mode or the use of a different time-dependence of the focusing is straightforward.

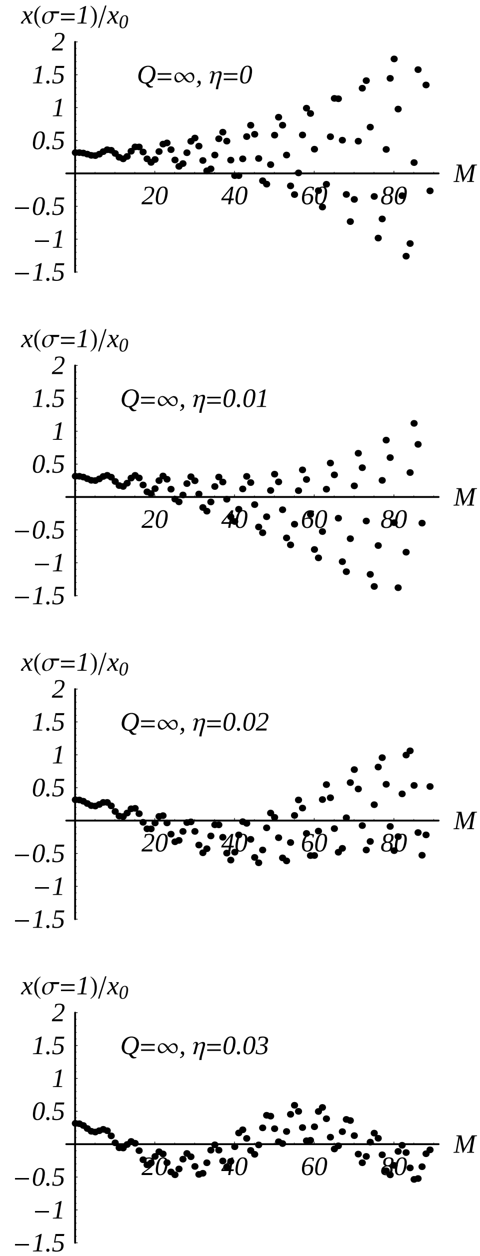

Figure 1: Normalized lateral displacement of a finite train of point-like bunches at the exit of a nominal linear collider. See Table 1 for the choice of parameters and Eq. (24) for the definition of η.

SUMMARY

This paper presents a formalism to address analytically cumulative beam breakup in linear accelerators with time-dependent parameters, such as energy chirp or rf focusing. It allows, in principle, direct calculation, at any time and location, of the transverse displacement of beams of arbitrary current distribution. When applied to a collider-like accelerator, the analytical results reproduce exactly the results of numerical simulations that were done previously.

While we assumed here constant injection offsets and a perfectly aligned accelerator, time-dependent offsets and misalignment of the cavities and focusing elements, as well as acceleration, can be included in this formalism and will be presented in another publication.

REFERENCES

[1] J. R. Delayen, Phys. Rev. ST Accel. Beams **6**, 084402 (2003)

[2] J. R. Delayen, Phys. Rev. ST Accel. Beams **7**, 074402 (2004)

[3] V. Balakin, A. Novokhatsky, and V. Smirnov, *Proc. 12th Conf. High Energy Accel*, Fermilab, 1983, p. 119

[4] C. L. Bohn and K.-Y. Ng, Phys. Rev. Lett. **85**, 984 (2000)

[5] C. L. Bohn and K.-Y. Ng, *Proc. XX International Linac Conference*, Monterey, CA, p. 31 (2000)