A 3D SELF-CONSISTENT, ANALYTICAL MODEL FOR LONGITUDINAL PLASMA OSCILLATION IN A RELATIVISTIC ELECTRON BEAM

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Abstract

Longitudinal plasma oscillations are becoming a subject of great interest for XFEL physics in connection with LSC microbunching instability [1] and certain pump-probe synchronization schemes [2]. In the present paper we developed the first exact analytical treatment for longitudinal oscillations within an axis-symmetric, (relativistic) electron beam, which can be used as a primary standard for benchmarking space-charge simulation codes. Also, this result is per se of obvious theoretical relevance as it constitutes one of the few exact solutions for the evolution of charged particles under the action of self-interactions.

INTRODUCTION

Longitudinal space-charge oscillations have been treated only from an electrodynamical viewpoint, or using limited one-dimensional models: in this paper we report a fully self-consistent solution to the initial value problem for the evolution of a relativistic electron beam under the action of its own fields in the (longitudinal) direction of motion. The beam is accounted for any given radial dependence of the particle distribution function. For a more detailed description of our work and references see [1]. An initial condition is set so that the beam, which is assumed infinitely long, is modulated in energy and density at a given wavelength. When the amplitude of the modulation is small enough the evolution equation can be linearized. An exact solution can be found in terms of an expansion in (self-reproducing) propagating eigenmodes.

Our theoretical findings constitute one of the few exact solutions known up to date to the problem of particles evolving under the action of their own fields. Yet, particle accelerator and FEL physics make large use of simulation codes to deal with space-charge fields, and these codes are benchmarked against exact solutions of Poisson equation only; recently partial attempts based on one-dimensional theory (which can only give some incomplete result) have been made to benchmark them against some analytical model accounting for the system evolution. We claim that our findings can be used as a standard benchmark for any space-charge code from now on. Our results are of relevance to an entire class of practical problems arising in state-of-the-art FEL technology when (optically) modulated electron beams are feed into an FEL (optical seeding or certain two-color pump-probe schemes): given the parameters of the system, plasma oscillations turn out to be an effect to be accounted for. It is also important to mention the relevance of plasma oscillation theory in the understanding of practical issues like longitudinal space-charge instabilities in high-brightness linear accelerators which may lead to beam microbunching and break up.

[1]E. Saldin et al. Longitudinal Spacs Charge Driven Michrobunching instability in TTF linac, TESLA-FEL-2003-02, May 2003

[2]J. Feldhaus et al. Two-color FEL amplifier for femtosecond-resolution pump-probe experiments with GW-scale X-ray and optical pulses DESY 03-091, July 2003

THEORY

We describe longitudinal plasma waves in a relativistic electron beam assuming that transverse coordinates enter as parameters in the description of fields and particle distribution. Our beam is initially modulated at some wavelength λ_m, in density and energy. It is natural to define the phase $\psi = \omega_m (z/v_z(\mathcal{E}_0) - t)$, where $v_z(\mathcal{E}_0) \sim c$ is the longitudinal electron velocity at the nominal beam kinetic energy $\mathcal{E}_0 = (\gamma-1)mc^2$, $\omega_m = 2\pi v_z/\lambda_m$, t is the time and z the longitudinal abscissa. We operate in energy-phase variables (P, ψ), P being the deviation from the nominal energy.

A small energy deviation P is assumed; then the equations of motion for our system can be interpreted as Hamilton canonical equations corresponding to the Hamiltonian $H(\psi, P, z) = e \int d\psi E_z + \omega_m P^2/(2c\gamma_z^2 \mathcal{E}_0)$. The bunch density distribution is then represented by $f = f(\psi, P, z; \boldsymbol{r}_\perp)$. Linearization of the evolution equation for f is possible when $f(\psi, P, z; \boldsymbol{r}_\perp)_{|z=0} = f_0(P; \boldsymbol{r}_\perp) + f_1(\psi, P, z; \boldsymbol{r}_\perp)_{|z=0}$, f_0 being the unperturbed solution of the evolution equation with $f_1 \ll f_0$ for any value of dynamical variables or parameters. Moreover we assume $f_0(P; \boldsymbol{r}_\perp) = n_0(\boldsymbol{r}_\perp) F(P)$, where the local energy spread function $F(P)$ is normalized to unity. The initial modulation can be written as a sum of density and energy modulation terms: $f_1(\psi, P, z; \boldsymbol{r}_\perp)_{|z=0} = f_{1d}(\psi, P; \boldsymbol{r}_\perp) + f_{1e}(\psi, P; \boldsymbol{r}_\perp)$ where $f_{1d}(\psi, P, z; \boldsymbol{r}_\perp) = a_{1d}(\boldsymbol{r}_\perp) F(P) \cos(\psi)$ and $f_{1e}(\psi, P, z; \boldsymbol{r}_\perp) = a_{1e}(\boldsymbol{r}_\perp) dF/dP \cos(\psi + \psi_0)$. Here ψ_0 is an initial (relative) phase between density and energy modulation. Finally it is convenient to define complex quantities $\tilde{f}_{1d} = a_{1d} F$, and $\tilde{f}_{1e} = a_{1e}(dF/dP) e^{i\psi_0}$ so that $f_{1_{|z=0}} = (\tilde{f}_{1d} + \tilde{f}_{1e}) e^{i\psi} + CC$. Further definition of $\tilde{E}_z = \tilde{E}_z(z; \boldsymbol{r}_\perp)$ in such a way that $E_z = \tilde{E}_z e^{i\psi} + \tilde{E}_z^* e^{-i\psi}$ allows one to write the Vlasov equation linearized in $\tilde{f}_1$:

$$\frac{\partial \tilde{f}_1}{\partial z} + i \frac{\omega_m P}{c\gamma_z^2 \mathcal{E}_0} \tilde{f}_1 - e\tilde{E}_z \frac{\partial f_0}{\partial P} = 0\,. \qquad (1)$$

Let us now introduce the longitudinal current density $j_z(z; \boldsymbol{r}_\perp) = -j_0(\boldsymbol{r}_\perp) + \tilde{j}_1 e^{i\psi} + \tilde{j}_1^* e^{-i\psi}$, where $j_0(\boldsymbol{r}_\perp) \simeq$

$ecn_0(\boldsymbol{r}_\perp)$ and $\tilde{j}_1 \simeq -ec\int_{-\infty}^{\infty} dP \tilde{f}_1$. From Eq. (1) follows

$$\tilde{j}_1 = -ec\int_{-\infty}^{\infty} dP \left(a_{1d}F + a_{1e}\frac{dF}{dP}e^{i\psi_0}\right) e^{-i\frac{\omega_m P z}{c\gamma_z^2 \mathcal{E}_0}} -ej_0\int_0^z dz'\left[\tilde{E}_z \int_{-\infty}^{\infty} dP \frac{dF}{dP} e^{i\frac{\omega_m P}{c\gamma_z^2\mathcal{E}_0}(z'-z)}\right] . \quad (2)$$

Next we present the equation for the electric field $\tilde{E}_z$ which, coupled with Eq. (2), will describe the system evolution in a self-consistent way.

Starting with the inhomogeneous Maxwell equation for the z-component of the electric field, passing to complex quantities and assuming that the envelope of fields and currents vary slowly enough over the z coordinate (this simply means that we can neglect retardation effects) we have

$$\nabla_\perp^2 \tilde{E}_z - \frac{\omega_m^2 \tilde{E}_z}{\gamma_z^2 c^2} = \frac{4\pi i\omega_m}{\gamma_z^2 c^2}\tilde{j}_1 , \quad (3)$$

which forms, together with Eq. (2), a self-consistent description for our system.

Combining Eq. (2) with Eq. (3) and using properly normalized quantities we obtain an integro-differential equation for the field evolution:

$$\hat{\nabla}_\perp^2 \hat{E}_z - q^2\hat{E}_z = iq^2\int_{-\infty}^{\infty} d\hat{P}\left(\hat{a}_{1d}\hat{F} + \hat{a}_{1e}\frac{d\hat{F}}{d\hat{P}}\right)e^{-i\hat{P}\hat{z}} -iq^2 S_0\int_0^{\hat{z}} d\hat{z}'\left[\hat{E}_z\int_{-\infty}^{\infty} d\hat{P}\frac{d\hat{F}}{d\hat{P}}e^{i\hat{P}(\hat{z}'-\hat{z})}\right] . \quad (4)$$

Definitions of naturally normalized quantities in Eq. (4) are as follows: $\hat{\boldsymbol{r}} = \boldsymbol{r}_\perp/r_0$, $\hat{E}_z = \tilde{E}_z/E_0$, $q = k_m r_0/\gamma_z$, $\hat{P} = P/(\rho\mathcal{E}_0)$, $\hat{a}_{1d} = -eca_{1d}/J_0$, $\hat{a}_{1e} = -ece^{i\psi_0}a_{1e}/(J_0\rho\mathcal{E}_0)$, $\hat{z} = \Lambda_P z$; $\hat{F}(\hat{P})$ is normalized to unity and S_0, the transverse profile function of the beam, obeys $S_0(\mathbf{0}) = 1$. Parameters are the typical transverse size of the beam r_0 , $J_0 = I_0\left[\int S(\boldsymbol{r}_\perp/r_0)d\boldsymbol{r}_\perp\right]^{-1}$, $E_0 = 4\pi J_0/\omega_m$ (where I_0 is the beam current), the plasma wave number $\Lambda_P = \left[4I/(I_A r_0^2\gamma\gamma_z^2)\right]^{1/2}$ ($I_A = mc^3/e$ being the Alfven current), $\rho = \Lambda_p\gamma_z^2/k_m$. Moreover the rms energy spread $\langle(\Delta\mathcal{E})^2\rangle$ can be measured by the dimensionless parameter $\hat{\Lambda}_T^2 = \langle(\Delta\mathcal{E})^2\rangle/\rho^2\mathcal{E}_0^2$ and the dimensionless current densities can be written as $\hat{j}_0 = j_0/J_0 \equiv S_0(\boldsymbol{r}_\perp/r_0)$ and $\hat{j}_1 = \tilde{j}_1/J_0$.

An equivalent description of the system evolution in terms of $\hat{j}_1$ can be obtained using the following result:

$$\hat{E}_z = -\frac{iq^2}{2\pi}\int d\hat{\boldsymbol{r}}_\perp^{(\boldsymbol{s})}\hat{j}_1 K_0\left(q \mid \hat{\boldsymbol{r}}_\perp - \hat{\boldsymbol{r}}_\perp^{(\boldsymbol{s})}\mid\right) , \quad (5)$$

where K_0 is the modified Bessel function of the second kind. Then, substitution in Eq. (2) and use of normalized quantities yield:

$$\hat{j}_1 = \int_{-\infty}^{\infty} d\hat{P}\left(\hat{a}_{1d}\hat{F} + \hat{a}_{1e}\frac{d\hat{F}}{d\hat{P}}\right)e^{-i\hat{P}\hat{z}} +\frac{iq^2}{2\pi}S_0\int_0^{\hat{z}} d\hat{z}'\left[\int d\hat{\boldsymbol{r}}_\perp^{(\boldsymbol{s})}\hat{j}_1 K_0\left(q \mid \hat{\boldsymbol{r}}_\perp - \hat{\boldsymbol{r}}_\perp^{(\boldsymbol{s})}\mid\right) \times\int_{-\infty}^{\infty} d\hat{P}\frac{d\hat{F}}{d\hat{P}}e^{i\hat{P}(\hat{z}'-\hat{z})}\right] . \quad (6)$$

Eq. (4) is particularly suitable for analytical manipulations, while Eq. (6) can be used better in case of a numerical approach.

MAIN RESULT

After introduction of the Laplace transform of $\hat{E}_z$, $\bar{E}(p,\hat{\boldsymbol{r}}_\perp)$, with $Re(p) > 0$, it follows from Eq. (4) that

$$\mathcal{L}\bar{E} = f \quad \text{with :} \quad (7)$$

$$\mathcal{L} = \hat{\nabla}_\perp^2 + \hat{g}(\hat{\boldsymbol{r}}_\perp, p) , \quad (8)$$

$$f(\hat{\boldsymbol{r}}_\perp, p) = iq^2\left(\hat{D}_0\hat{a}_{1d} + \hat{D}\hat{a}_{1e}\right) , \quad (9)$$

$$\hat{g}(\hat{\boldsymbol{r}}_\perp, p) = -q^2(1 - i\hat{D}S_0) , \quad (10)$$

$$\hat{D}_0 = \int_{-\infty}^{\infty} d\hat{P}\frac{\hat{F}}{p+i\hat{P}} , \quad \hat{D} = \int_{-\infty}^{\infty} d\hat{P}\frac{d\hat{F}/d\hat{P}}{p+i\hat{P}} \quad (11)$$

with the boundary conditions $\bar{E} \longrightarrow 0$ for $|\hat{\boldsymbol{r}}_\perp| \longrightarrow \infty$ and $\partial\bar{E}/\partial\hat{\boldsymbol{r}}_\perp \longrightarrow 0$ for $|\hat{\boldsymbol{r}}_\perp| \longrightarrow \infty$. Solution is found when we find a Green function $\bar{G}$ such that $\bar{E} = \int d\hat{\boldsymbol{r}}'_\perp \bar{G}(\hat{\boldsymbol{r}}_\perp, \hat{\boldsymbol{r}}'_\perp)f(\hat{\boldsymbol{r}}'_\perp)$.

Assuming, without prove, completeness and discreteness of the spectrum of $\mathcal{L}$ (we ascribe to alternative theoretical approaches and numerical techniques the assessment of the validity region of this assumption) we can expand $\bar{G}$ using the eigenfunction of $\mathcal{L}$ defined by $\mathcal{L}\Psi_j = \Lambda_j\Psi_j$ thus getting

$$\bar{E} = \sum_j \frac{\Psi_j(\hat{\boldsymbol{r}}_\perp)}{\Lambda_j}\int d\hat{\boldsymbol{r}}'_\perp \Psi_j(\hat{\boldsymbol{r}}'_\perp)f(\hat{\boldsymbol{r}}'_\perp) . \quad (12)$$

To find $\hat{E}_z$ we use the inverse Laplace transformation and we perform the integration analytically with the help of Jordan lemma. We write results in a general form, but this method is straightforward only in the case of a cold beam $\hat{F} = \delta(\hat{P})$ that will be the only one considered here. Our final result is written as follows:

$$\hat{E}_z(\hat{z}, \hat{\boldsymbol{r}}_\perp) = \sum_j u_j\Phi_j(\hat{\boldsymbol{r}}_\perp)e^{\lambda_j\hat{z}} , \quad (13)$$

$$u_j = \frac{\int d\hat{\boldsymbol{r}}'_\perp \Phi_j(\hat{\boldsymbol{r}}'_\perp)f(\hat{\boldsymbol{r}}'_\perp, \lambda_j)}{\left[\int d\hat{\boldsymbol{r}}'_\perp\left(\frac{\partial g}{\partial p}\right)\Psi_j^2\right]_{p=\lambda_j}} . \quad (14)$$

The modes Φ_j are not orthogonal. Appropriate initial conditions can be chosen to obtain a single propagating mode at *fixed* values of j, namely:

$$\frac{\hat{a}_{1e}}{\hat{a}_{1d}} = -i\lambda_j \quad , \quad \hat{a}_{1d} = (\hat{\nabla}_\perp^2 - q^2)\Phi_j . \quad (15)$$

From now on we will deal with case of an axis-symmetric beam described using a cylindrical coordinate system $(\hat{r}, \phi, \hat{z})$, with obvious meaning of symbols. It is convenient to discuss azimuthal harmonics of $\hat{j}_1$, $\hat{E}_z$ and f which will be indicated with $\hat{j}_1^{(n)}(z, \hat{r})$, $\hat{E}_z^{(n)}(\hat{z};\hat{r})$ and $f^{(n)}(\hat{r}, p)$. Our results Eq. (13) and Eq. (14) take the simpler form:

$$\hat{E}_z^{(n)}(\hat{z}, \hat{r}) = \sum_j u_{nj}\Phi_{nj}(\hat{r})e^{\lambda_j^{(n)}\hat{z}} , \quad (16)$$

$$u_{nj}(\hat{r}) = \frac{\int_0^\infty d\hat{r}'\hat{r}'\Phi_{nj} f^{(n)}(\hat{r}',\lambda_j^{(n)})}{\left[\int_0^\infty d\hat{r}'\hat{r}'\left(\frac{\partial g}{\partial p}\right)\Psi_{nj}^2\right]_{p=\lambda_j^{(n)}}} . \quad (17)$$

We give here some explicit calculations for several profile cases.

Stepped profile - In this case $S_0 = 1$ for $\hat{r} < 1$ and $S_0 = 0$ for $\hat{r} \geq 1$. Putting $\alpha_j^2 = -q^2(1 + 1/\lambda_j^{(n)\ 2})$ we obtain the eigenvalue equation:

$$\alpha_j J_{n+1}(\alpha_j) K_n(q) - q K_{n+1}(q) J_n(\alpha_j) = 0 . \quad (18)$$

It turns out that $\lambda_j^{(n)}$ are imaginary and such that $-1 < Im(\lambda_j^{(n)}) < 1$. The solution for the evolution equation is:

$$\hat{E}_z^{(n)}(\hat{z},\hat{r}) = \left\{ \begin{array}{ll} \sum_j u_{nj} J_n(\alpha_j \hat{r}) e^{\lambda_j^{(n)}\hat{z}} & \hat{r} < 1 \\ \sum_j u_{nj} \frac{J_n(\alpha_j)}{K_n(q)} K_n(q\hat{r}) e^{\lambda_j^{(n)}\hat{z}} & \hat{r} \geq 1 \end{array} \right. , \quad (19)$$

$$u_{nj} = \frac{K_n(q)\int_0^1 d\xi J_n(\alpha_j \xi)\xi f^{(n)}(\xi)}{J_n(\alpha_j)\frac{d}{dp}\left[\alpha J_{n+1}(\alpha)K_n(q) - qK_{n+1}(q)J_n(\alpha)\right]_{p=\lambda_j^{(n)}}} , \quad (20)$$

where $\alpha_j^2 = -q^2(1 + 1/p^2)$.

Parabolic profile - In this case $S_0(\hat{r}) = 1 - k_1^2\hat{r}^2$ for $\hat{r} < 1/k_1$ and $S_0 = 0$ for $\hat{r} \geq 1/k_1$. Solution for the homogeneous problem defined by $\mathcal{L}$ can be found in literature (see [1] for references). We can use that solution in order to solve our eigenvalue problem, and to write the expressions for the eigenfunctions Ψ_{nj} to be inserted in Eq. (13). Let us introduce the following notations: $\mu^2 = i\hat{D}q^2 - \Lambda_j^{(n)}$, $\delta^2 = i\hat{D}K_1^2$, $d^2 = \Lambda_j^{(n)}$, $\epsilon = (n+1)/2 - \mu^2/(4\delta)$. After some calculation we find:

$$\Psi_{nj}(\hat{r}) = \left\{ \begin{array}{ll} \hat{r}^n e^{-\delta\hat{r}^2/2}\ {}_1F_1(\epsilon, n+1, \delta\hat{r}^2) & \hat{r} < 1 \\ e^{-\delta/2}\ {}_1F_1(\epsilon, n+1, \delta)\frac{K_n(d\hat{r})}{K_n d} & \hat{r} \geq 1 \end{array} \right. . \quad (21)$$

where ${}_1F_1$ is the confluent hypergeometric function, and the eigenvalue equation analogous of Eq. (18) is now

$$\delta K_n(d)\left[2\epsilon(n+1)^{-1}\ {}_1F_1(\epsilon+1, n+2, \delta) - {}_1F_1(\epsilon, n+1, \delta)\right] + dK_{n+1}(d)\ {}_1F_1(\epsilon, n+1, \delta) = 0 . \quad (22)$$

Multilayer method approach - An arbitrary gradient axisymmetric profile can be approximated by means of a given number of stepped profiles, or layers, superimposed one to the other. Results for the stepped profile case can be then used to construct an algorithm to deal with any profile (see [1] for more details).

ALGORITHM FOR NUMERICAL CALCULATIONS

The linear regime assumption is not too restrictive but it would be interesting to provide a solution for the full problem. As a first step towards this goal we present here a numerical solution of the evolution equation in the case of an axis-symmetric beam, that we cross-checked with our main result, Eq. (13). In order to build a numerical solution it turns out convenient to make use of Eq. (6).

After some manipulations Eq. (6) yields:

$$\frac{d^2\hat{j}_1^{(n)}}{d\hat{z}^2} = -q^2 S_0 \int_0^1 d\hat{r}'\hat{r}' G^{(n)}\hat{j}_1^{(n)} , \quad (23)$$

where

$$G^{(n)}(\hat{r},\hat{r}') = \left\{ \begin{array}{ll} I_n(q\hat{r})K_n(q\hat{r}') & \hat{r} < \hat{r}' \\ I_n(q\hat{r}')K_n(q\hat{r}) & \hat{r} > \hat{r}' \end{array} \right. , \quad (24)$$

Eq. (23) is to be considered together with proper initial conditions for $\hat{j}_1$ and its z-derivative at $z = 0$. The interval $(0,1)$ can be then divided into an arbitrary number of parts so that Eq. (23) is transformed in a system of the same number of 2nd order coupled differential equations to be solved numerically. This gave us the solution of the evolution problem in terms of the beam current. Then we calculated back $\hat{E}_z$ and we compared obtained results with Eq. (13) for different choices of transverse profiles. The real field E_z should be recovered but all relevant information is included in $Re(\hat{E}_z)$. We calculated $Re(\hat{E}_z)$ as a function of $\hat{z}$ and $\hat{r}$ in several cases (see [1]). Comparison with the Runge-Kutta integration program were performed (see [1]) and gave a perfect agreement. Finally we actually verified that selecting a single mode by fixing appropriate initial conditions as described in Eq. (15) is possible (see, again, [1]).

CONCLUSIONS

In this paper paper we presented one of the few self-consistent analytical solutions for a system of charged particles under the action of their own electromagnetic fields. Namely, we considered a relativistic electron beam under the action of space-charge at given initial conditions for energy and density modulation and we developed a fully analytical, three-dimensional theory of plasma oscillations in the direction of the beam motion in the linear regime. We specialized the general method to the important cases of stepped and parabolic transverse profiles, which are among the few analytically solvable situations. In particular, the stepped profile case could be used to develop a semi-analytical technique to solve the evolution problem for the field using an arbitrary transverse shape. We also developed an algorithm able to solve the evolution problem in terms of the beam currents. Numerical and analytical or semi-analytical solutions for the fields were then compared and gave a perfect agreement. Finally we showed how to build up initial conditions in such a way that a single mode is excited and propagates through and we checked our prescription by setting up particular initial conditions and looking at the propagation of various eigenmodes.

REFERENCES

[1] G. Geloni, E. Saldin, E. Schneidmiller and M. Yurkov, DESY 04-112, see http://xxx.lanl.gov/abs/physics/0407024