Interpretation of Particle Motion in a Circular Accelerator as Diffraction of Light

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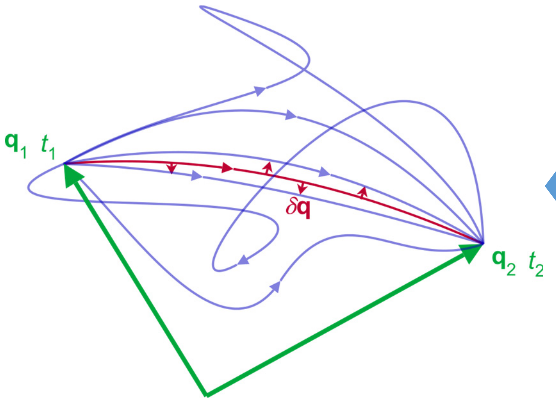
Introduction

Optico-Mechanical analogy

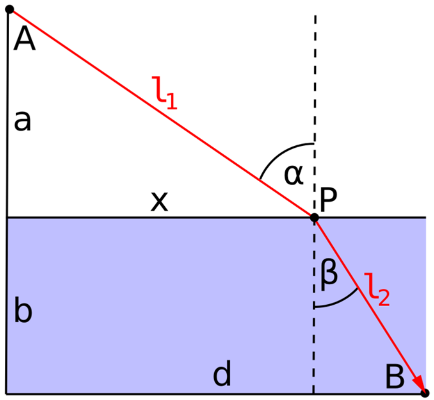
About 200 years ago, W.R. Hamilton derived one of the most important principles in classical mechanics, called “Principle of least action”, inspired by the analogy between optics and mechanics.

Figures taken from Wikipedia

Principle of least action



Fermat’s Principle



William Rowan Hamilton (1805 – 1865)

Revisit the analogy

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Forced harmonic oscillator interpreted as diffraction of light

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We investigate a simple forced harmonic oscillator with a natural frequency varying with time. It is shown that the time evolution of such a system can be written in a simplified form with Fresnel integrals, as long as the variation of the natural frequency is sufficiently slow compared to the time period of oscillation. Thanks to such a simple formulation, we found that a forced harmonic oscillator with a slowly varying natural frequency is essentially equivalent to diffraction of light.

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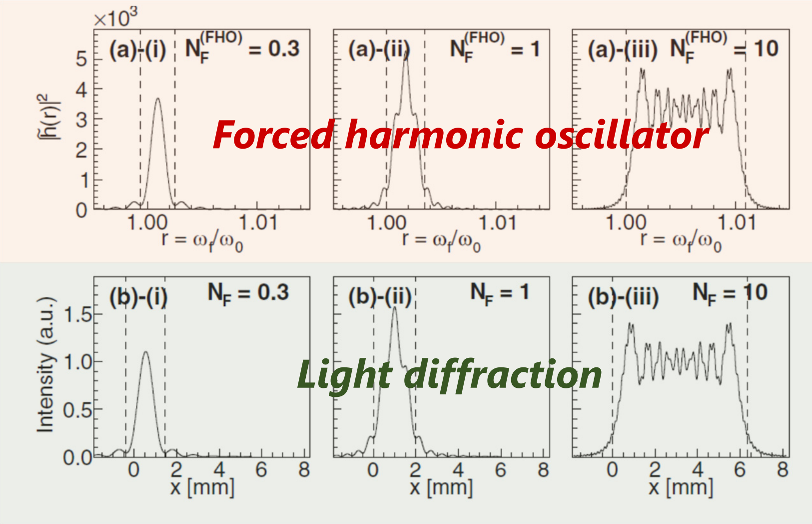
Frequency response of forced harmonic oscillator mimics an intensity pattern of light diffraction from a single-slit.



In this talk, we present the analogy between the particle motion in a circular accelerator and the light diffraction.

How the idea came about?

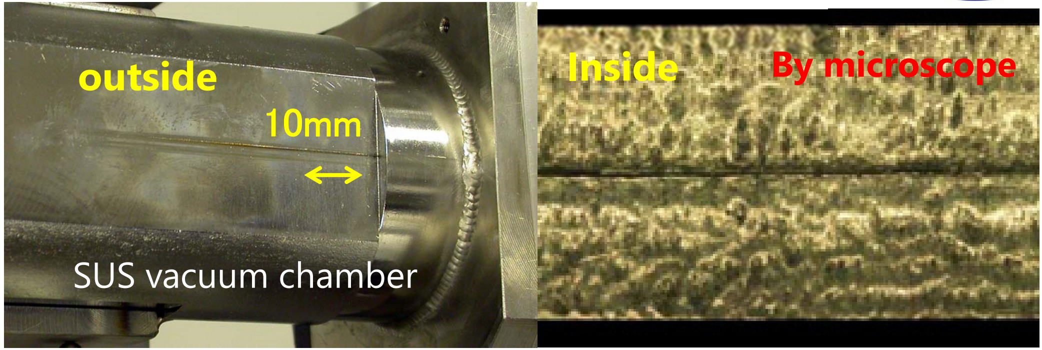
- The idea was born when we were analyzing the transverse motion of an aborted electron beam for the design of an aborted beam handling system in diffraction-limited rings – especially in the ring for the 3GeV Light Source Project (Sendai, Japan).
- In the 3GeV light source ring, when the stored beam is dumped, not only the RF power for acceleration is switched off, but also a sinusoidal time-varying kick with a constant frequency is applied to the aborted beam by a beam shaker to reduce the beam density.
- Since the aborted beam gradually loses its energy by synchrotron radiation, the betatron tune changes gradually according to the chromaticity, and consequently a resonant condition also changes every moment.
- A simple question – *What is the most effective shaker's frequency?* – naturally led us to modeling the aborted beam motion, and eventually we found "the analogy to light diffraction from a single slit".

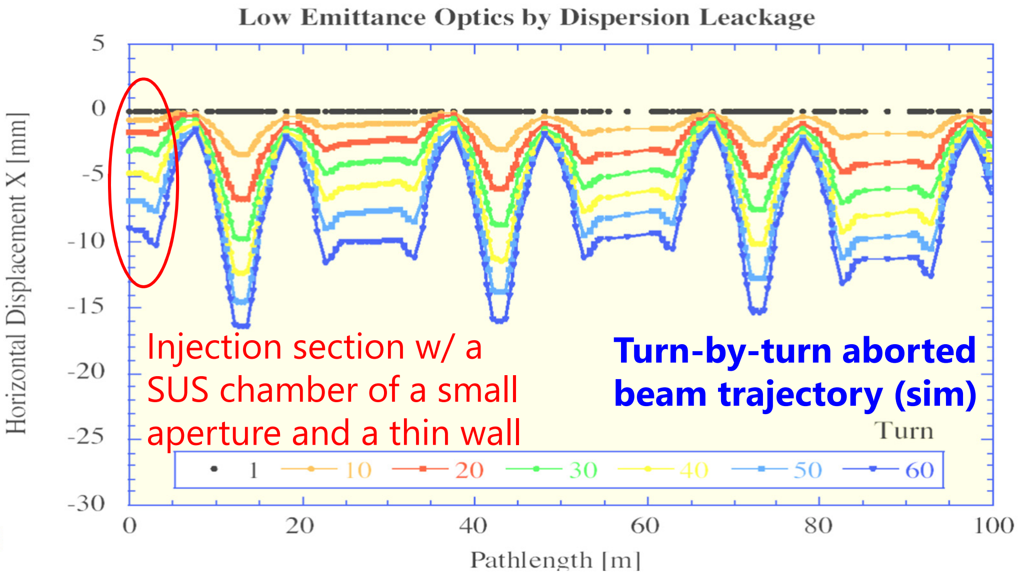
Outline

- ■Background
  - ✓Importance of safe beam abort in diffraction-limited rings (DLRs)
  - ✓Abort beam handling in the 3GeV light source ring
- ■Formulation of an aborted beam motion with an external sinusoidal kick
- ■Application to the case of the 3GeV light source ring
- ■Analogy between electron motion and light diffraction
- ■Summary

Importance of safe beam abort in DLRs

- Concern for high density/current beam in DLRs:
  - Some cares must be taken when beams are dumped
- Accident in the SPring-8 ring:
  - Aborted electron beams (8GeV, 100mA, ε_0=3.4nmrad) melted a vacuum chamber made of SUS
  → Vacuum leakage occurred
- Tracking simulation:
  - Unexpected peak of higher-order dispersion caused by the change in optics settings
  - SUS chamber of a small aperture and a thin (0.7mm) wall at the injection section
  - Concentration of heat load by electron beams on the SUS chamber





For the 3GeV ring, please visit Nishimori-san's talk on 6/16(Thu)

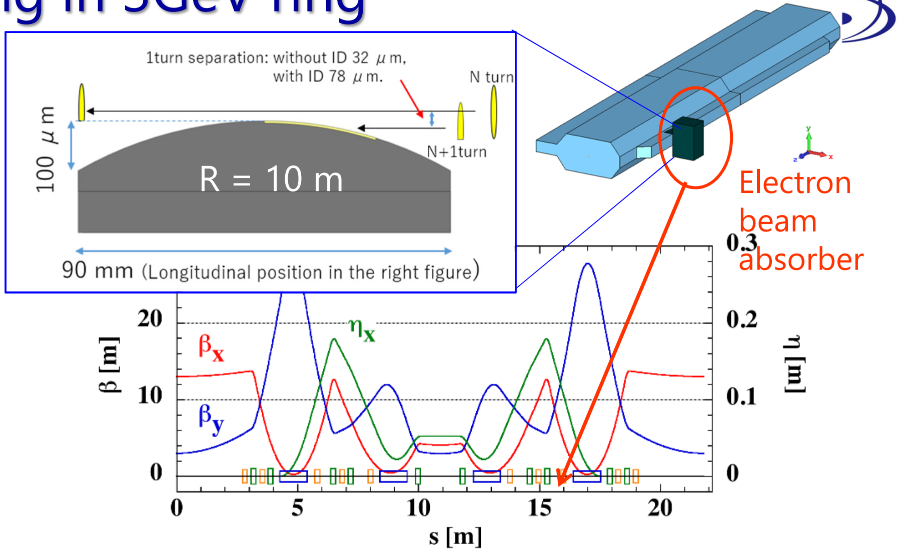
Abort beam handling in 3GeV ring

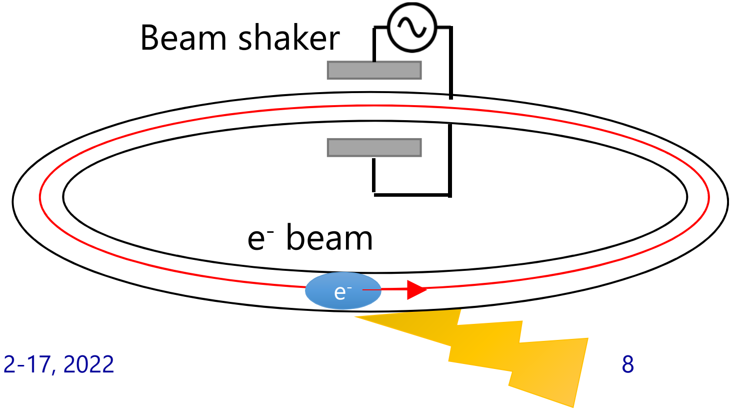
Two countermeasures for the protection of vacuum chambers from high-density/current beams:

- **Electron beam absorber**
  - ✓ Made of graphite (low-Z, high melting point)
  - ✓ Placed near a dispersion peak of each cell
  - ✓ Curved (R=10m) surface to scatter off electrons and distribute the heat load
- **Beam shaker**
  - ✓ Activated shortly after switching off the RF power (beam abort)
  - ✓ Apply sinusoildal-patterned kicks to the beam to spread the beam vertically and reduce the beam density.

$$\boxed{\theta(t) = \theta_0 \sin\left(\frac{2\pi\nu_f t}{T_{rev}}\right)}$$

θ_0: Maximum kick (= 2 µrad)
ν_f: Shaker's frequency in terms of tune
T_{rev}: Revolution period (= 1.164 µs)





A simple question

What is the most effective shaker's frequency?

- Simple forced harmonic oscillator:

$$\ddot{x} + \omega_0^2 x = F_0 \sin(\omega_f t + \phi_0)$$

Resonant condition

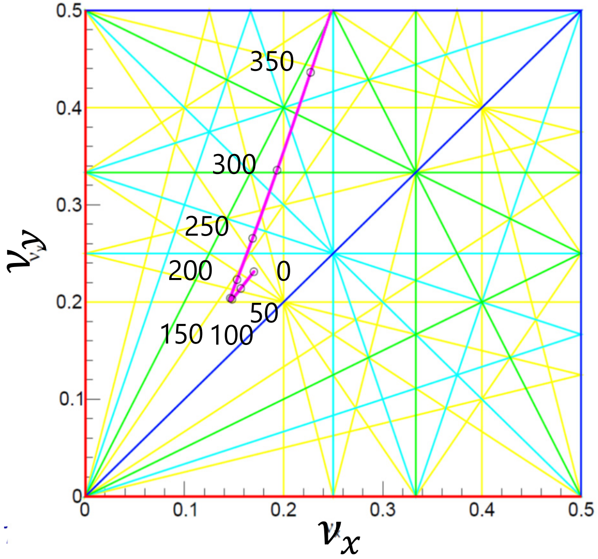
$$\boxed{\omega_f = \omega_0}$$

- Aborted beam with an external sinusoidal force:

$$\ddot{x} + \omega^2(t) x = F_0 \sin(\omega_f t + \phi_0)$$

Time-dependent due to energy loss by synchrotron rad

$$\omega_f = ?$$

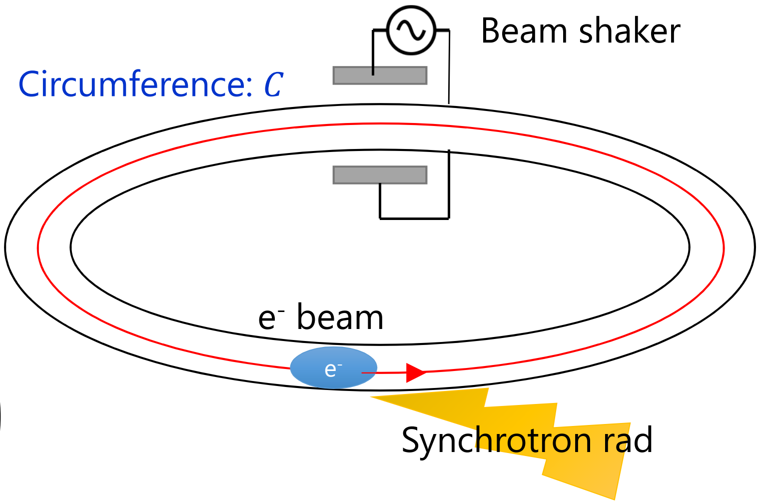


Formulation

- Suppose that an electron undergoes a sinusoidal time-varying kick every turn from the beam shaker at $s = s_f$ shortly after the RF is switched off ($t = 0$).
- Transverse motion of an electron is determined by:

$$\boxed{\frac{d^2 z}{ds^2} + K(s)z = F(s)}$$

Basic equation



with an external force by beam shaker:

$$F(s) = \delta(s - s_f) F_0 \cos\left({}^{2\pi\nu_f s}\!/_{C} \right)$$

Shaker's frequency in terms of "tune"

Formulation (cont'd)

- Introducing the following variables (Courant-Snyder transformation):

$$u \equiv \frac{z}{\sqrt{\beta}} \qquad \nu \equiv \frac{1}{2\pi}\oint \frac{ds}{\beta} \qquad \phi \equiv \frac{1}{\nu}\int \frac{ds}{\beta}$$

- The basic equation becomes "a quite familiar form":

$$\boxed{\frac{d^2u}{d\phi^2} + \nu^2 u = \bar{F}(\phi)}$$

← Same form as the equation of motion for a forced harmonic oscillator

where

$$\bar{F}(\phi) = \nu^2 \beta^{\frac{3}{2}} \left|\frac{ds}{d\phi}\right|^{-1} \sum_{0 \le \phi_{f,n} \le \phi} \delta(\phi - \phi_{f,n})\, F_0 \cos(\nu_f \phi + \phi_0)$$

$$\equiv \sum_{0 \le \phi_{f,n} \le \phi} \bar{F}_0 \cos(\nu_f \phi + \phi_0) \qquad \bar{F}_0 \equiv \nu\sqrt{\beta} F_0$$

Chromatic effect

- Since the RF is switched off, the electron loose its energy gradually through synchrotron radiation (Left top fig).
- So the betatron tune also varies gradually according to the chromaticity (Right bottom fig).
- In our model, such an effect is taken into account as "chromatic effects":

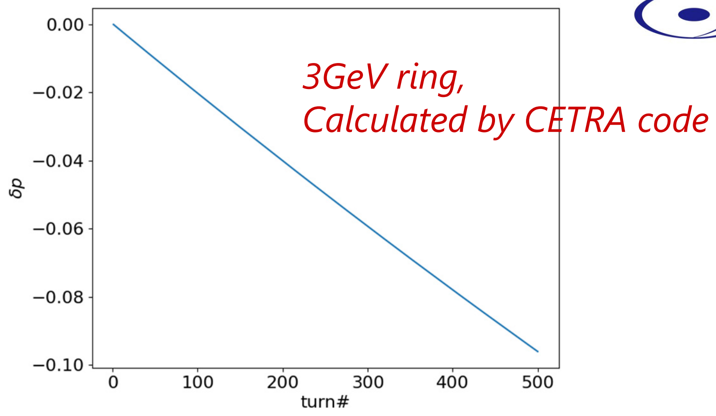
$$\frac{d^2u}{d\phi^2}+\nu^2(\phi)u=\bar{F}(\phi)$$

- NOTE: since the linear chromaticity takes a small positive value, "adiabatic condition" holds in general:

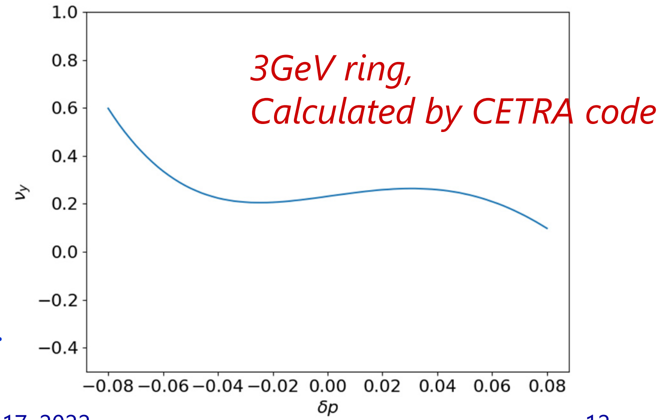
$$|\dot{\nu}(\phi)| \ll \nu^2(\phi) \qquad |\ddot{\nu}(\phi)| \ll \nu^3(\phi)$$

*Betatron tune varies **very slowly** compared to betatron oscillation.*

Momentum δp as a function of turn#



Non-linear chromaticity (Vertical)



Green's function

- General solution for an inhomogeneous differential equation:

$$\mathcal{L}y(x) = f(x) \quad \Rightarrow \quad y(x) = \int dx' G(x,x') f(x') \qquad \text{with } \mathcal{L}G(x,x') = \delta(x'-x)$$

G: Green's function

- Green's function can be obtained using two independent solutions of the corresponding homogeneous equation [$\mathcal{L}a_i(x) = 0$]:

$$G(x,x') = \frac{\begin{vmatrix} a_1(x') & a_2(x') \\ a_1(x) & a_2(x) \end{vmatrix}}{W(a_1,a_2)(x')}$$

W: Wronskian

- In the present case, a Green's function can be obtained under the adiabatic condition:

$$G(\phi,\phi') = \frac{-i}{2\sqrt{\nu(\phi)\nu(\phi)}} \exp\left[i \int_{\phi'}^{\phi} d\chi \nu(\chi) \right] + \mathrm{c.c.}$$

Transverse motion of an aborted electron

- A particular solution of the basic equation is written as:

$$u(\phi) = \underbrace{\frac{i\bar{F}_0}{4\sqrt{\nu(\phi)}} \exp\left[-i\int_0^{\phi} \nu(\chi)d\chi\right]}_{\text{Harmonic oscillator with frequency modulation}} \times \underbrace{h(\phi;\nu_f)}_{\text{Frequency response to the beam shaker}} + \mathrm{c.c.}$$

- Envelope function (response function) is defined as:

$$h(\phi;\ \nu_f) = \frac{1}{2\pi}\int_0^{\phi} \frac{d\phi'}{\sqrt{\nu(\phi')}} \exp\left[i\left\{\int_0^{\phi'} \nu(\chi)d\chi - \nu_f - \phi_0\right\}\right]$$

Application to the 3GeV ring

- Our simple model was applied to the 3GeV ring, and the results were compared with those of tracking simulation, *CETRA*, which performs a symplectic integration based on the Hamiltonian.
- For simplicity, a perfect ring was assumed; i.e. there is no error magnetic field in both cases.
- Radiation loss in the tracking simulation was calculated using expected values.
- In our model, the chromatic effect was implemented as a fifth order polynomial function, and each coefficient was determined by fitting the *CETRA* result (Right fig).

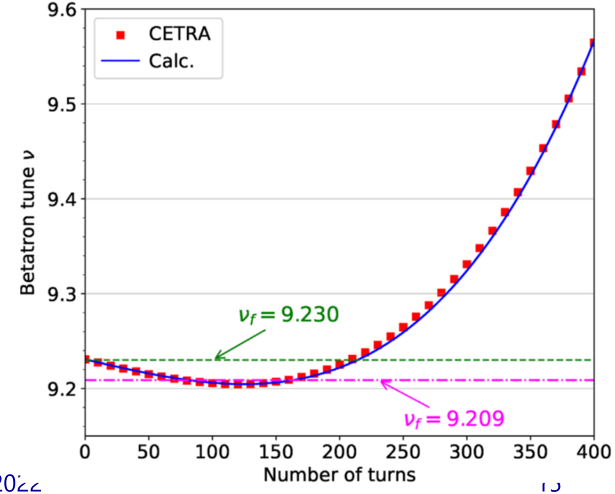
$$\nu(\delta) = \nu_0 + \xi_1\delta + \xi_2\delta^2 + \xi_3\delta^3 + \xi_4\delta^4 + \xi_5\delta^5$$

$$\delta = -n\Delta$$

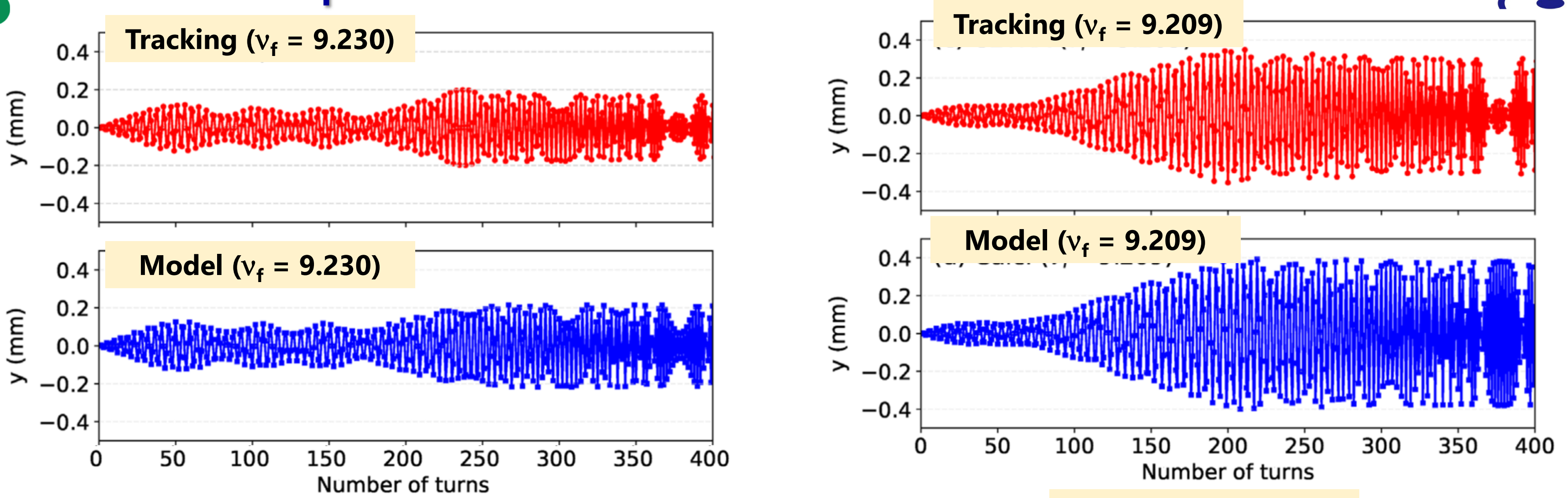
$$\Delta = U/E_0$$

U: Energy loss per turn

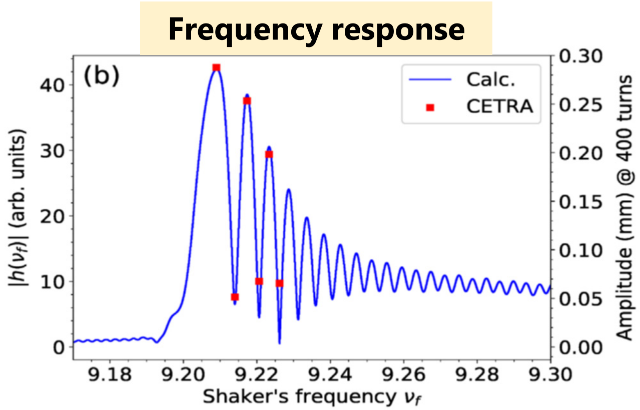
E_0: Reference energy



Simple model vs Tracking simulation



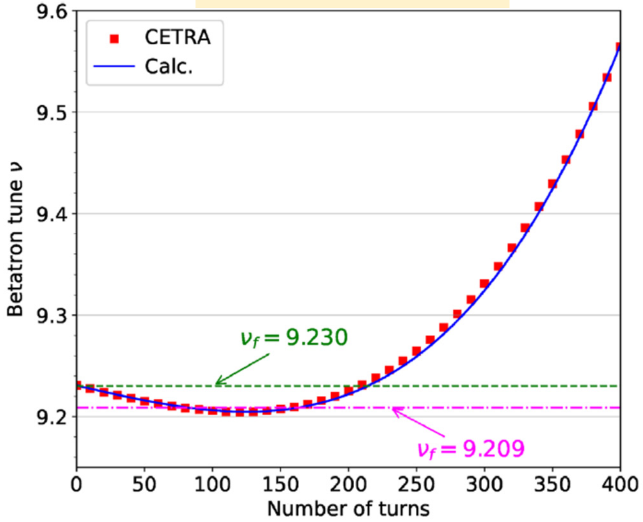
- Up to ~230 turns, our model is in excellent agreement with the CETRA tracking.
- After that, we see some discrepancies probably due to:
  - The change in path length due to radiation loss, causing a timing shift of receiving a kick
  - The deterioration of the adiabatic condition due to non-linearities of chromaticity
- Nevertheless, our model serves well for quick estimate of frequency response (see right bottom fig.).

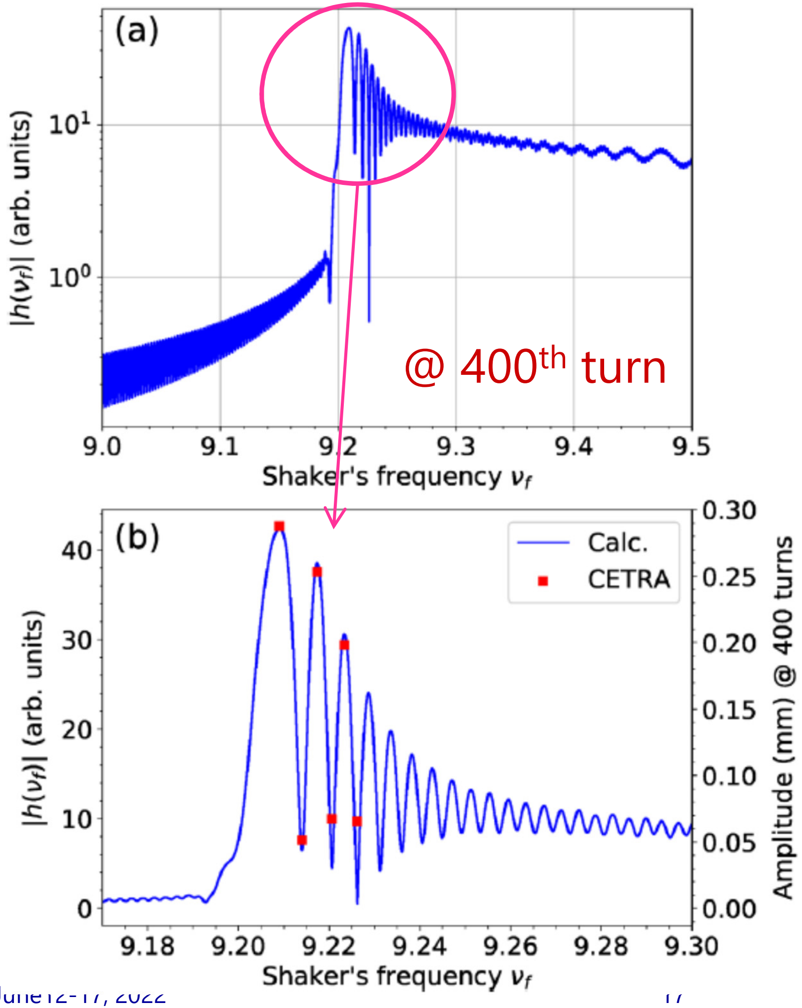


Frequency response

- Gross structure is well understood:
  - ➢ How many times the aborted beam crosses the resonant condition?
  - ➢ How fast the aborted beam crosses the resonant condition?
- What is the origin of the structure like a diffraction pattern?

Tune excursion





Linearly-varying tune

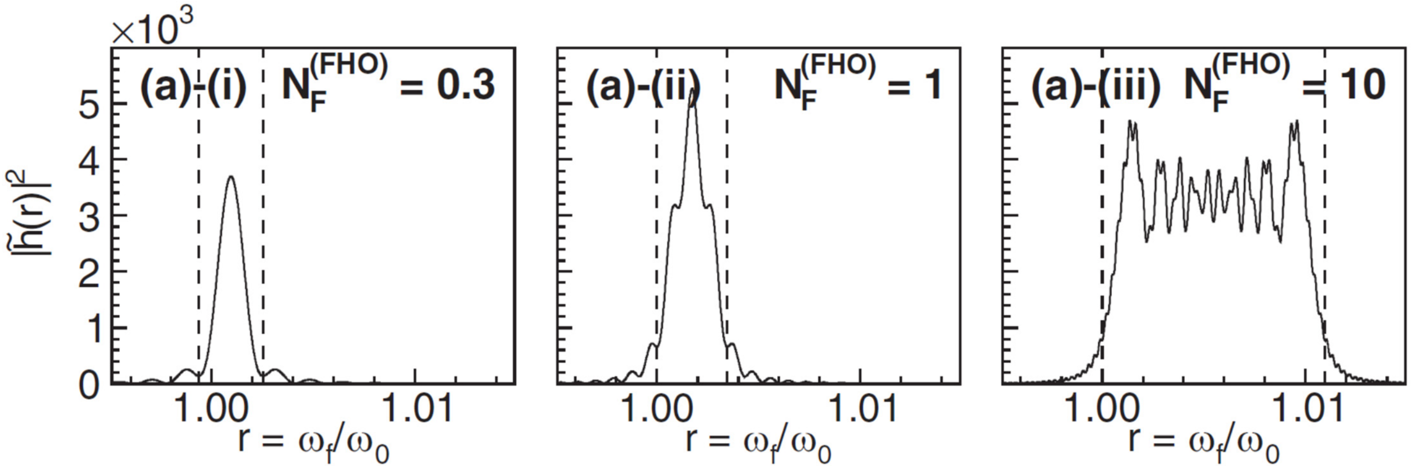
- Response function for linearly-varying tune:

$$h(\phi;\nu_f) = \frac{1}{\sqrt{2\nu_0\xi_1\Delta}}\exp\left[-i\left\{\phi_0 - \frac{\pi(\nu_0-\nu_f)^2}{\xi_1\Delta}\right\}\right] \times [\{C(u_2)-C(u_1)\} - i\{S(u_2)-S(u_1)\}]$$

$$\left\{\boxed{\begin{aligned} C(u) &= \int_0^u \cos\left(\frac{\pi}{2}v^2\right)dv \\ S(u) &= \int_0^u \sin\left(\frac{\pi}{2}v^2\right)dv \end{aligned}}\right.$$

← Fresnel integrals

$$\left\{\begin{aligned} u_1 &= -\frac{\sqrt{2}(\nu_0-\nu_f)}{\sqrt{\xi_1\Delta}} \\ u_2 &= u_1 + \sqrt{2\xi_1\Delta} \end{aligned}\right.$$



← *Looks like an intensity pattern for single-slit diffraction!!*

Analogy to light diffraction

■ Aborted electron beam:

Frequency response

$$h(\phi;\nu_f) \propto \int_0^{\phi} \exp\left[i\left\{2\pi(\nu_0-\nu_f)\left(\frac{\phi}{2\pi}\right)\left(\frac{\phi'}{\phi}\right) - \pi\xi_1\Delta\left(\frac{\phi}{2\pi}\right)^2\left(\frac{\phi'}{\phi}\right)^2\right\}\right] d\phi'$$

■ Single-slit diffraction:

Electric field on the screen

$$E(x) \propto \int_0^{2a} \exp\left[i\left\{-4\pi N_F \frac{x}{a}\left(\frac{\xi}{2a}\right) - 4\pi N_F\left(\frac{\xi}{2a}\right)^2\right\}\right] d\xi$$

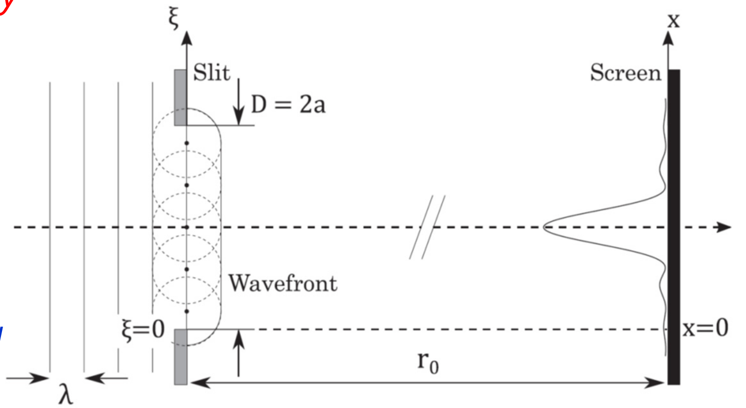
$N_F \equiv \dfrac{a^2}{\lambda r_0}$: Fresnel number

Both phase functions change "quadratically" on the integral variable!!

- Relations connecting two phenomena:

$$\boxed{\begin{aligned} 2\pi\left(\frac{\phi}{2\pi}\right)(\nu_f-\nu_0) &\Leftrightarrow 2N_F\frac{x}{a} \\ \xi_1\Delta\left(\frac{\phi}{2\pi}\right)^2 &\Leftrightarrow 4N_F \end{aligned}}$$

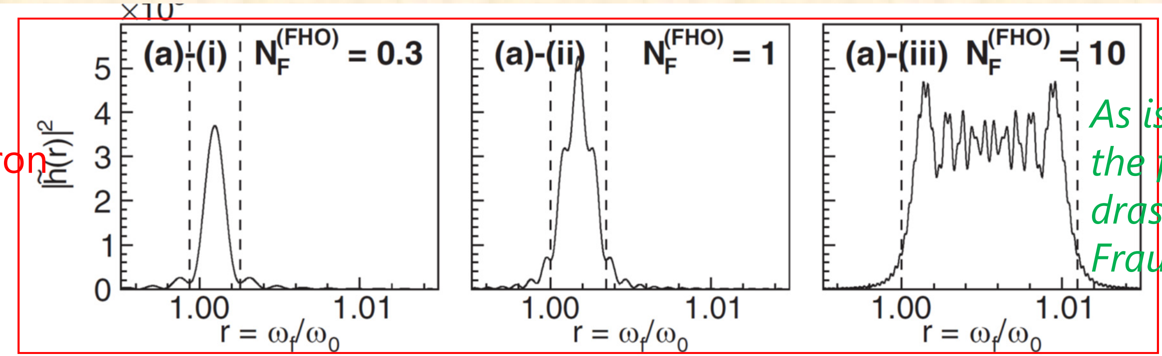
← Fresnel number can be also defined for aborted beam



Frequency response and diffraction pattern

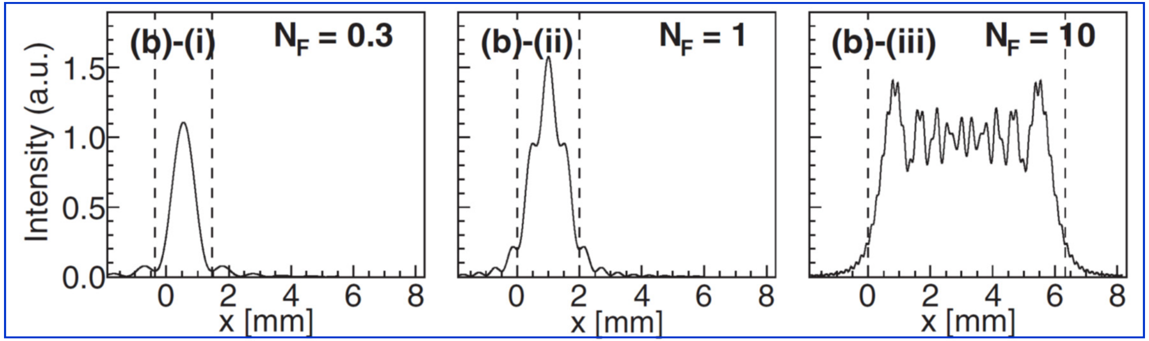
In the light diffraction theory, a Fresnel number N_F is often used to characterize a diffraction pattern:
- $N_F \ll 1$: Fraunhofer (far-field) diffraction occurs. A diffraction pattern is a Fourier transform of the slit aperture.
- $N_F \gtrsim 1$: Fresnel (near-field) diffraction occurs. A diffraction pattern is just a shadow of the slit aperture.

Aborted electron beam



As is the light diffraction case, the frequency response changes drastically at $N_F = 1$ from Fraunhofer-type to Fresnel-type.

Single-slit diffraction



Correspondence relation

Aborted electron beam	Single-slit diffraction
Shaker's frequency, ν_f ※	Position on the screen, x
Betatron tune of the electron, ν	Position inside the slit, ξ
Total change in betatron tune, $\xi_1 \Delta \frac{\phi}{2\pi}$	Slit aperture size, $2a$
Phase slippage btwn betatron oscillation and sinusoidal patterned kick	Quadratic change in optical path length
A quantity, $\tilde{N}_F \equiv \frac{\xi_1 \Delta}{4} \left(\frac{\phi}{2\pi}\right)^2$	Fresnel number, $N_F \equiv \frac{a^2}{\lambda r_0}$

※ Strictly speaking, it is not shaker's frequency itself, but the deviation of shaker's frequency w.r.t. the initial betatron tune.

Application (1)

- Can be used as a quick estimator for optimal shaker's frequencies
- In light diffraction, diffraction patterns are well understood:

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Fraunhofer regime ($N_F \ll 1$)

$$\bar{x}_{\mathrm{min/max}} = a \pm \frac{2a}{4N_F}$$

Principal diffraction peak

Fresnel regime ($N_F \gtrsim 1$)

$$\bar{x}_{\mathrm{min/max}} = a \pm a$$

Just a shadow of slit aperture

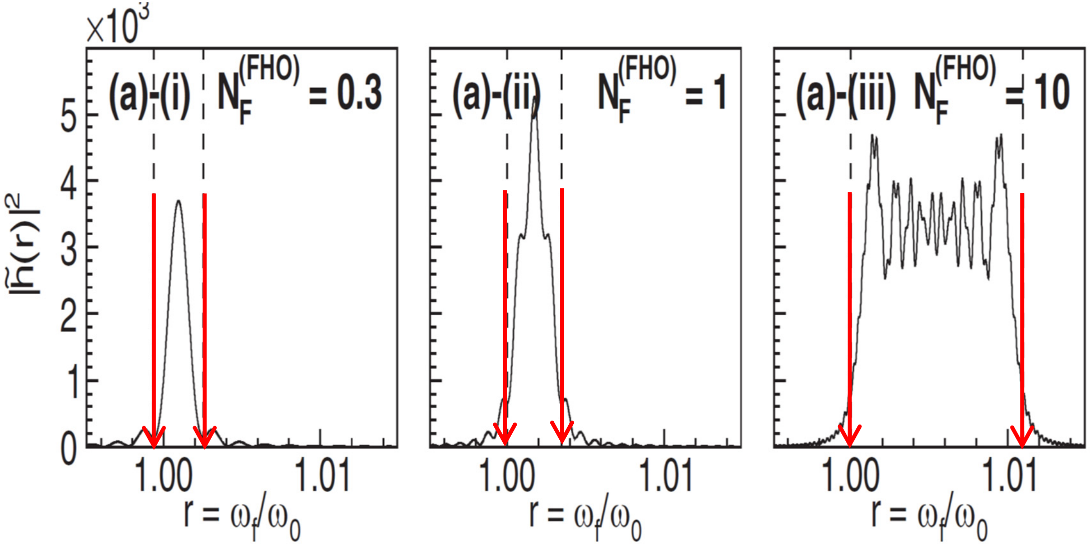
- Shaker's frequency for beam abort:

Fraunhofer regime ($N_F \ll 1$)

$$\boxed{\nu_f^{\mathrm{min/max}} = \nu\left(\frac{\phi}{2}\right) \pm \frac{2\pi}{\phi}}$$

Fresnel regime ($N_F \gtrsim 1$)

$$\boxed{\nu_f^{\mathrm{min/max}} = \nu\left(\frac{\phi}{2}\right) \pm \frac{1}{2}\xi_1 \Delta \frac{\phi}{2\pi}}$$



Application (2)

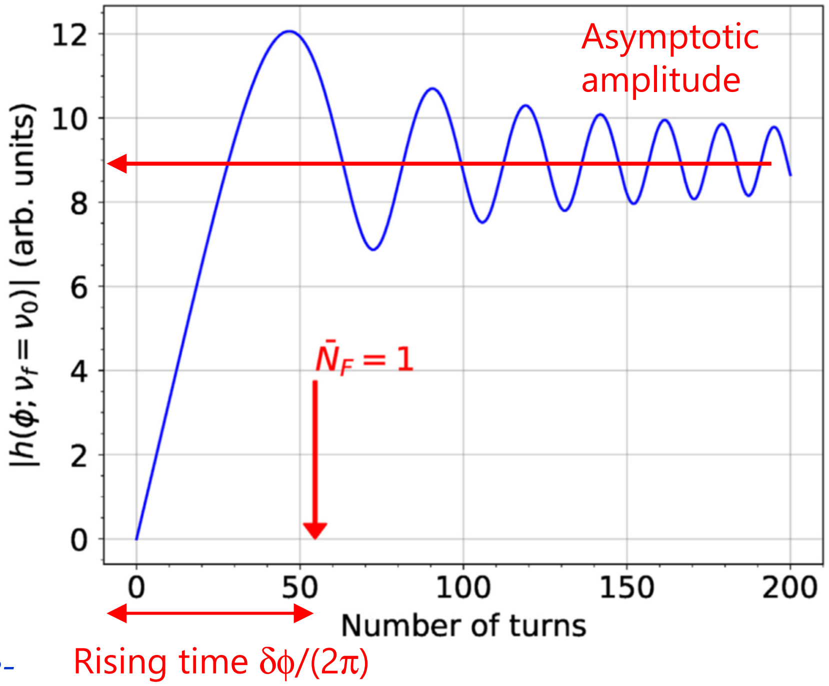
- Applicable to the analysis of resonance-crossing phenomena
- Time development of oscillation amplitude behaves like knife-edge diffraction (i.e., Fresnel diffraction from a straight edge):

$$h(\phi;\nu_f=\nu_0)=\frac{e^{-i\phi_0}}{\sqrt{2\nu_0\xi_1\Delta}}\left[C\left(\sqrt{2\xi_1\Delta}\frac{\phi}{2\pi}\right)-iS\left(\sqrt{2\xi_1\Delta}\frac{\phi}{2\pi}\right)\right]$$

- Using our knowledge of light diffraction:

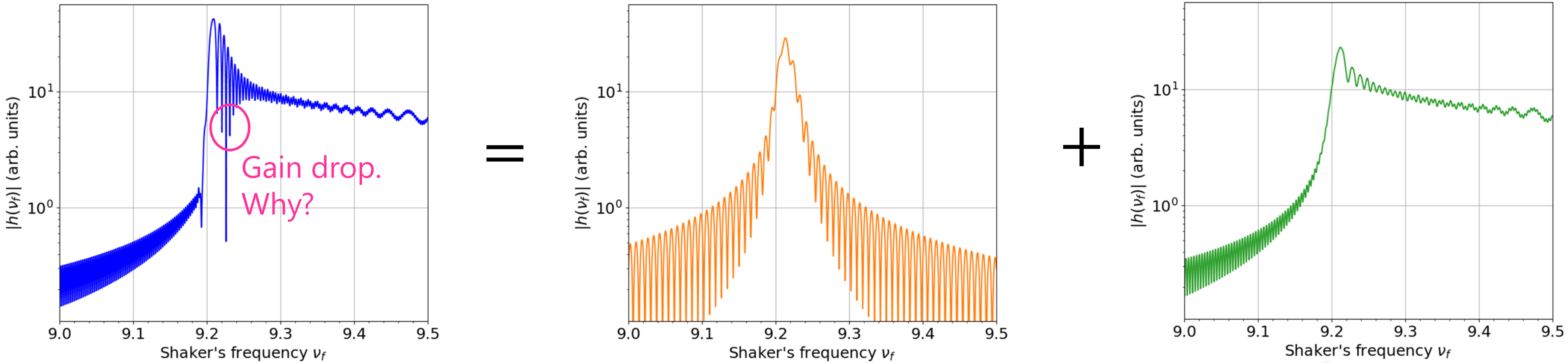
Rising time: $\boxed{\left(\frac{\delta\phi}{2\pi}\right)=\frac{1}{\sqrt{\xi_1\Delta}}}$

Asymptotic amplitude: $\boxed{|h|_\infty=\frac{1}{2\sqrt{\nu_0\xi_1\Delta}}}$

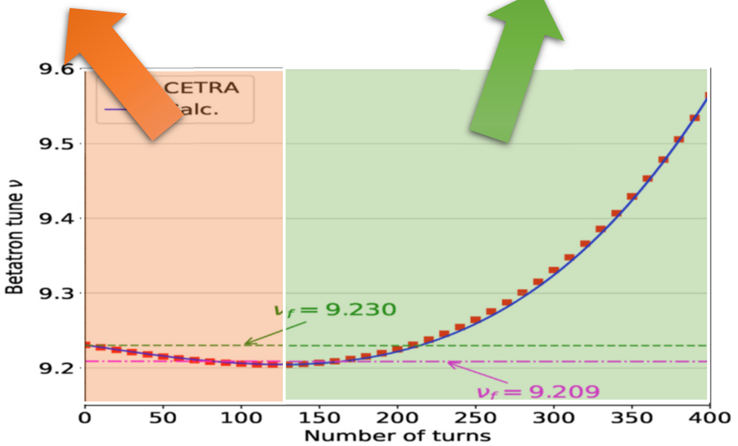


Consistent with the well-known fact that "the effect of resonance-crossing is inversely proportional to the square-root of crossing-speed".

A closer look: Frequency response



- Can be interpreted as a further superposition of two diffraction patterns with different $\overline{\mathrm{N}}_{\mathrm{F}}$.
- → Gain drop around $\nu_f \sim 9.23$ is attributed to destructive interference between the two diffraction pattern.



Summary

- We found that the behavior of an aborted electron is quite similar to light diffraction, in designing a safe beam abort system for diffraction-limited rings.
- This stems from the phase slippage between the betatron oscillation and the sinusoidal-patterned force, caused by a slowly-varying betatron tune, which mimics the parabola approximation of optical path length in Fresnel's light diffraction theory.
- With this analogy, we can predict the aborted beam motion in a very straightforward way.
- We expect that our idea will provide a simple and intuitive approach to the analyses of resonance-crossing phenomena, which is of great concern in designing a ring-type accelerator.

Thank you for your attention!