ISBN: 978-3-95450-227-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2022-WEPOMS018

MINIMUM EMITTANCE GROWTH DURING RF-PHASE SLIP

S. Koscielniak†, TRIUMF, Vancouver B.C. Canada

Abstract

This paper is concerned with finding operations consistent with the absolute minimum emittance growth. The system is an RF bucket containing a bunch of hadrons in a synchrotron; and the operation performed is to sweep the RF phase. As a result, the bunch centroid moves from one value of position and momentum to another. For given start and end points, we shall find the ideal RF phase-slip time-variation that minimizes emittance growth of the bunch.

INTRODUCTION

Customarily, emittance is the phase space area, at one moment in time, occupied by an ensemble. Single-particle "emittance" is the area swept out during an oscillation cycle. Both are conserved quantities in Hamiltonian dynamics. The bounding single-particle emittance can be identified as surrogate for the ensemble (bounding) emittance. Typically the ensemble is contained by longitudinal phase-focusing and a transverse focusing channel. In nonlinear systems, some processes may introduce voids into the phase space, such that the bounding emittance effectively increases, even though the detailed occupied area is preserved. This phenomenon, known as "emittance growth", has been studied extensively – particularly for the case of variation of the strength of focusing parameters [1], but much less so for the case that the centre of focusing moves (as occurs during an RF phase sweep).

The area swept out during a cycle is a surrogate for occupied area; and the former is proportional to the Hamiltonian, H, for that phase space orbit. Thus we can use changes of the bounding value of H to predict emittance growth. In general, an RF sweep changes H of both the centroid and a general particle trajectory. Thus to minimize emittance growth, we must minimize the differential change of Hamiltonian between general particle and centroid: ($\Delta H - \Delta H_c$). Subscript c denotes centroid. The minimization is performed with respect to variation (choice) of the RF-phase-sweep time law for given end points (final and initial) of the centroid coordinates. There is a minor complication: an arbitrary sweep may either increase or decrease the oscillation amplitude (i.e. Hamiltonian) depending on the oscillation phase at the moments the particle encountered the perturbation. So we must select those bounding trajectories for which $\Delta H>0$, and oscillation amplitude increases.

In general, the bunch centroid does not follow an arbitrary RF phase sweep. The choice of sweep is strongly constrained by the condition that the centroid be in equilibrium at the centre of the RF bucket at start and end of the sweep. If not satisfied, the bunch centroid circulates within the bucket and there is growth of the area swept out by the bunch; and eventually the area is filled due to "filamentation". The primary constraint is that the sweep be completed in an integer number, *n*, of synchrotron oscillations of the centroid. Nevertheless there is typically a small residual oscillation because the momentum offset caused by the RF sweep does not accrue enough phase slip of the bunch to catch up to the RF phase. The residual (which scales as $1/n^2$) may either be accepted, or zeroed by making a "fast" RF-phase jump at start or end of the sweep. "Fast" means completed in a small fraction of a synchrotron oscillation. A phase jump is the cause of a jump in H, but jump values are cancelled out when the difference ($\Delta H - \Delta H_c$) is formed.

Ideally, the difference ($\Delta H - \Delta H_c$) would be minimized with respect to the choice of the RF phase sweep, f(t), according to the calculus of variations, in the manner of the brachistochrone problem. However, we have not found a suitable variational principle; and so have resorted to trial and error in the choice of f(t). The trial functions are: (1) linear ramp, (2) ½-sinusoid, (3) bi-quadratic, (4) dual-sinusoid; and (5) cubic – all ramped between t=0 and t=T.

The steps in the procedure are: first, compute matching conditions for the centroid; second compute ($\Delta H - \Delta H_c$) for the trajectories as perturbed by the sweep f(t). The result is parametrized by the initial amplitude and number of synchrotron oscillations for the sweep. In one case alone, linear f(t), all quantities may be calculated exactly in terms of the Jacobi elliptic functions; and these results used to benchmark other methods. For other sweeps, we may introduce a simplified (and artificial) Hamiltonian that facilitates computation of ($\Delta H - \Delta H_c$) and retains the frequency dispersion of the pendulum oscillator, but adopts a harmonic potential well. Results from these analytic methods will be compared to those from particle tracking.

ANALYSIS

The equation of motion is that of a pendulum oscillator with a moving pivot point f(t):

$$x'(t) = p\,(t) \text{ and } p'(t) = -\omega^2 \sin\,(f\,(t) + x\,(t)). \quad (1)$$

The change of Hamiltonian is:

$$\Delta H = \omega^2 \int_0^T f'(t) \sin\,(f\,(t) + x\,(t))\, dt. \quad (2)$$

The system is simplified by the transformations:

$$x(t) = \mathrm{x2}\,(t) - f\,(t) \;\&\; p(t) = \mathrm{p2}\,(t) - g\,(t). \quad (3)$$

Here *g(t)=df/dt* is the momentum offset that will generate precisely the phase slip *f(t)*. The motion equations become:

$$\mathrm{x2}'(t) = \mathrm{p2}\,(t) \text{ and } \mathrm{p2}'(t) = f''(t) - \omega^2 \sin\,(\mathrm{x2}\,(t)) \quad (4)$$

$$\text{and } \Delta H = \omega^2 \int_0^T g\,(t) \sin\,(\mathrm{x2}\,(t))\, dt. \quad (5)$$

The centroid matching conditions [2] are:

$$\Delta \mathrm{x2}_c = \int_0^T \mathrm{p2}_c\, dt = 0 \;\&\; \Delta \mathrm{p2}_c = -\omega^2 \int_0^T \mathrm{x2}_c\, dt = 0;$$

and cannot be satisfied unless T is an integer number of synchrotron oscillation periods.

† shane@triumf.ca

ISBN: 978-3-95450-227-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2022-WEPOMS018

Linear RF-phase Sweep

When g is a constant, as in the linear ramp $f(t)=(t/T)\Delta f$, the equations (4-5) are solved in terms of Jacobi elliptic functions (sn, cn ,dn) or (cd, sd, nd) with Jacobi amplitude parameter $m=k^2$. The Hamiltonian value is $H=2\omega^2 m$. Let $\sin(x2(u)/2) = k\,\mathrm{sn}\,(u + u0; m)$ and the change [3] $\Delta cn = cn(U+u_0;m) - cn(u_0;m)$. $U=\omega T$ is the accrued phase. Δcn describes a dipole oscillation. The normalized fractional change of Hamiltonian for the general trajectory is:

$$(\Delta H/H)\ (2\pi/\Delta f) = -\Delta cn/(n\,k). \qquad (6)$$

The initial phase u_0 and amplitude k are correlated by the disturbance f, and depend weakly on Δf; but we take them as independent. ΔH of the bunch is minimized when we set duration $U=2\pi\,n$. u_0(m) is chosen to find the largest ΔH on the bounding trajectory. Note: for the linear ramp and short bunches, ΔH does not fall as $1/n$ because the de-phasing (between different oscillation amplitudes) and Δcn is also proportional to n. Equation (6) may be used to benchmark particle tracking simulations and a simplified analysis that adopts a harmonic oscillator with an artificial amplitude-dependent frequency spread.

Simplified Equations

The equation of motion is that of a harmonic oscillator with an artificial frequency spread:

$$x'(t) = p\,(t) \ \text{ and } \ p'(t) = -\omega\,(a)^2(f\,(t) + x\,(t)). \quad (7)$$

Parameter a is the initial amplitude. ω(a) is the frequency dispersion of the pendulum. We adopt the same transformations as above. The initial Hamiltonian and change are:

$$H_0 = \tfrac{1}{2}a^2\,\omega(0)\,\omega(a),\ \Delta\mathrm{H} = \omega\,(a)^2\int_0^T g\,(t)x2\,(t)\,dt. \quad (8)$$

Equations (7) can be solved for x2 and p2 as particular integral and complementary function for arbitrary f(t); and ΔH calculated analytically. We do this for general particle and centroid, and form the difference $(\Delta H - \Delta H_c)$.

The centroid matching conditions depend on f(t). For the cases of (a) linear ramp & arbitrary n, and (b) bi-quadratic ramp and even n, no phase jump is needed. For the ½-sinusoid ramp, an initial jump $\Delta f/(4n^2-1)$ is required. Figure 1 shows matched centroid trajectories for the bi-quadratic ramp when Δf =1 radian and for n=2.

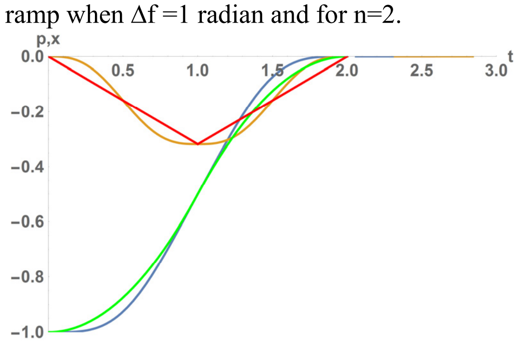

Figure 1: Response to bi-quadratic ramp. Curves are: green is –f(t); red is 2×g(t); blue is x(t); and gold is 2×p(t).

Benchmarking

We compare the normalized fractional change of Hamiltonian (ΔH/H)(2π/Δf) versus maximum oscillation amplitude X for the linear RF-phase ramp as calculated by three methods for a variety of sweep durations ωT=2nπ. Colour coding: n=1, blue; n=2, gold; n=3, olive; n=4 coral. See Figs. 2 through 4. For small amplitudes, the bounding trajectories are those emanating from the position axis (p=0); while for large amplitudes they emanate from the momentum axis (x=0).

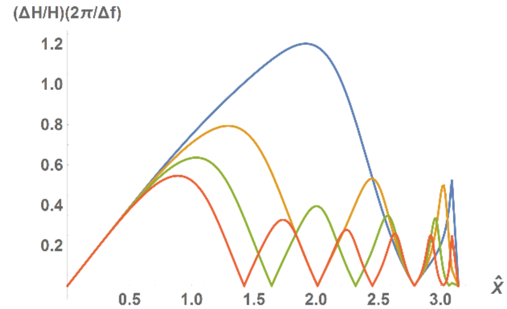

Figure 2: Theoretical ΔH from Jacobi functions (p=0).

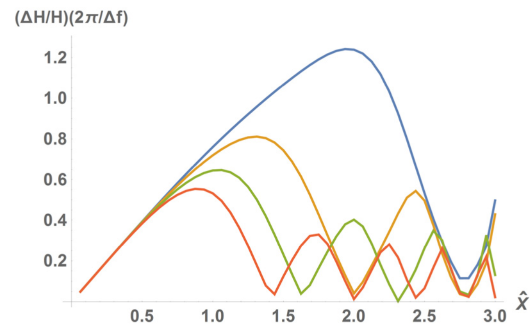

Figure 3: Simulated ΔH from particle tracking (p=0).

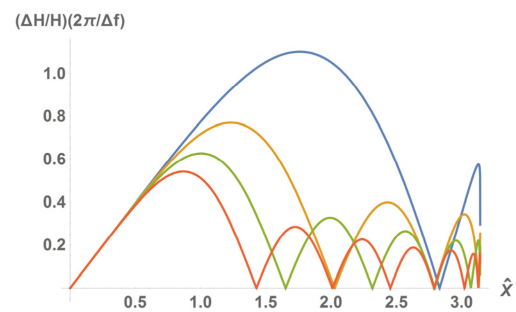

Figure 4: Predicted ΔH from simplified equations (p=0).

Figures 2 and 3 are in quantitative agreement, and Fig.4 in qualitative agreement. All plotted curves have zeros, and these are due to co-periodicity. E.g. the gold curve: one cycle of amplitude X=2 equals two cycles of the centroid.

Figure 5 shows the particle trajectories (emanating from p=0) used to compute ΔH for linear ramp with n=1 and Δf=1. The centroid motion is shown in red.

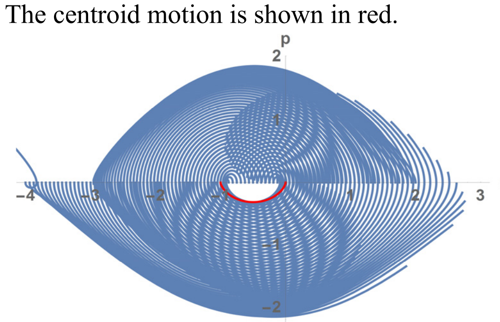

Figure 5: Particle trajectories for linear RF-phase sweep.

ISBN: 978-3-95450-227-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2022-WEPOMS018

Predictions & Confirmations

Figures 6-9 show the predicted normalized fractional change of Hamiltonian (for the bounding trajectory) for phase ramps completed in n= 1, 2, 3, 4 synchrotron periods. Figures 10-13 show corresponding results from particle tracking. The colour indicates the type of ramp f(t). Blue = linear ramp; orange = dual-sinusoid; magenta = ½-sinusoid; green = bi–quadratic; and red = cubic ramp.

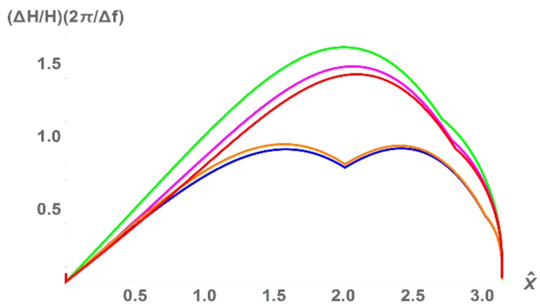

Figure 6: RF-phase ramps in 1 synchrotron oscillation.

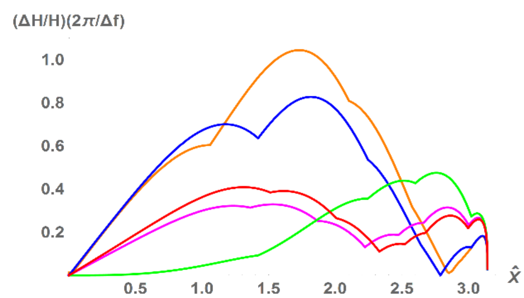

Figure 7: RF-phase ramps in 2 synchrotron oscillations.

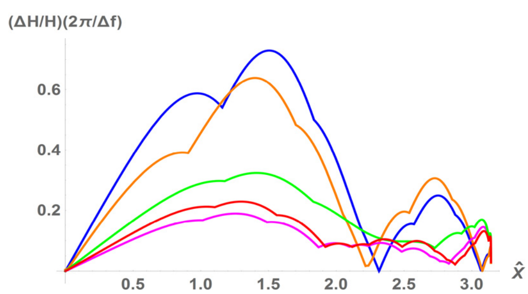

Figure 8: RF-phase ramps in 3 synchrotron oscillations.

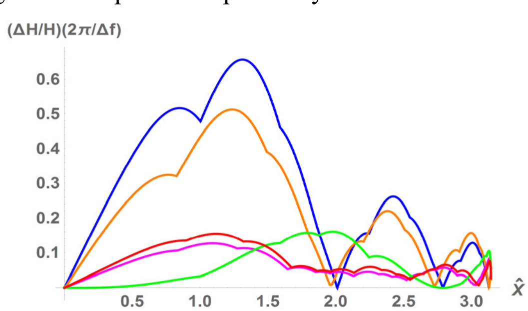

Figure 9: RF-phase ramps in 4 synchrotron oscillations. Conclusions: (1) "linear" is best for n = 1. Also: if linear ramp is used and amplitudes <0.5 radian, then no point using n>1. (2) "Bi-quadratic" is best for n = 2 and amplitudes < 2 radian. (3) "½-sine" is best for n = 3. (4) “Bi-quadratic” is best for n = 4 unless amplitudes < 1.5 radian.

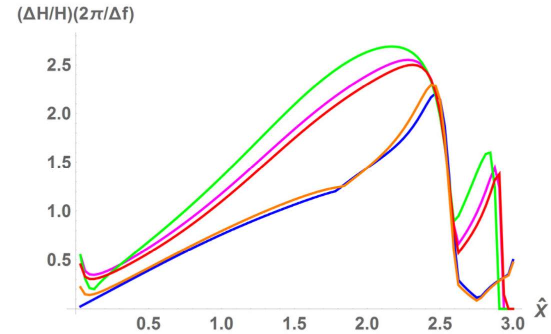

Figure 10: RF-phase ramps in 1 synchrotron oscillation.

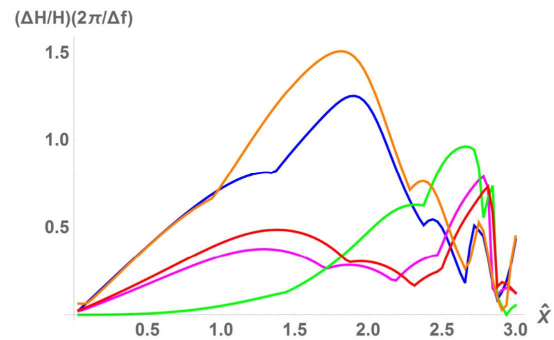

Figure 11: RF-phase ramps in 2 synchrotron oscillations.

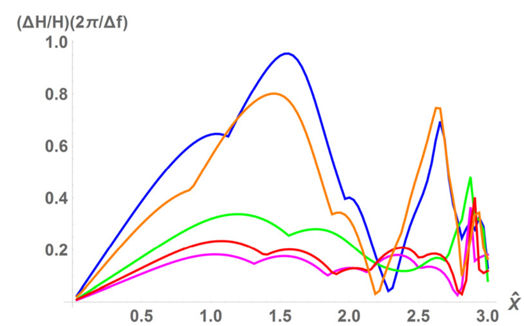

Figure 12: RF-phase ramps in 3 synchrotron oscillations.

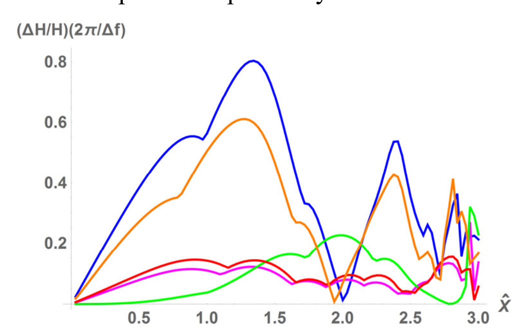

Figure 13: RF-phase ramps in 4 synchrotron oscillations.

CONCLUSION

We have introduced a method to determine conditions for minimum emittance growth. Predictions are in good agreement with particle tracking, particularly the ordering of the best ramp versus number of synchrotron oscillations.

ISBN: 978-3-95450-227-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2022-WEPOMS018

REFERENCES

[1] A.J. Lichtenberg, *Phase Space Dynamics of Particles,* Wiley, New York, 1969.

[2] S. Koscielniak, "When does a pendulum follow motion of its pivot?", in *Proc. Int. Workshop on Fixed Field Accelerators (FFA'20),* TRIUMF, Canada, Nov. 2020. https://ffa20.triumf.ca

[3] S. Koscielniak, "Minimizing longitudinal emittance growth during RF phase sweep" ?", in *Proc. Int. Workshop on Fixed Field Accelerators (FFA'21),* KURNS, Kyoto Japan, Sept. 2021. https://www.rri.kyoto-u.ac.jp/beam_physics_lab/ffa21/