Symplectic Tracking Through Field Maps

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Abstract

For studies of beam dynamics with complicated geometries of the fields, it is necessary to track particles using field maps, instead of an analytic representation of the fields which is typically not available. These field maps come about while designing elements such as realistic magnets or radiofrequency cavities, and represent the field geometry on a mesh in space. However, simple interpolation of the fields from the field maps does not guarantee that the resulting tracking scheme satisfies the symplectic condition. Here we present a general method to decompose the field-map potential as a sum of interpolating functions that produces, by construction, a symplectic integrator.

Core Concept —
Symplectic integrators, Poisson brackets, and the failures of interpolation

Most symplectic integrators use a split map approach, based on the Lie algebraic formalism, e.g.:

$$\mathcal{M}_{(s\to s+\Delta s)} \approx e^{-:H_0:\Delta s/2} e^{-:H_1:\Delta s} e^{-:H_0:\Delta s/2}$$

This requires exact evaluation of the Poisson brackets, which have partial derivatives of the coordinates. This means that a “kick” needs to be computed as the *exact* gradient of a scalar function.

$$p^f = p^i + \nabla_q V(q^i)\Delta s$$

For 2- and 3-D field maps, interpolating the fields does not guarantee the resulting kick is the exact gradient of a scalar function.

$$\vec{F} = \sum_{i,j,k} w_{i,j,k} \vec{f}_{i,j,k} \stackrel{?}{=} \nabla_q V(q)$$

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Symplectic integration using field maps requires an exactly differentiable interpolation between grid points in the field map.

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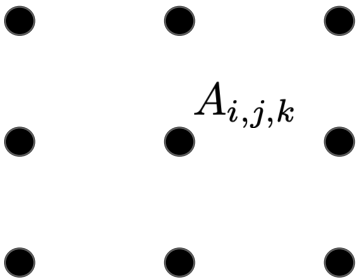
Exactly Differentiable Representations of Field Maps

Field maps know the vector potential at discrete points in space



Use locally differentiable basis functions to define interpolation scheme

$$A_{i,j,k} = \sum_{\sigma} a_{\sigma} \Psi_{\sigma}(\vec{r}_{i,j,k})$$

Take all required derivatives and integrals using local basis functions

$$\frac{\partial A}{\partial x}(\vec{r}) = \sum_{\sigma} a_{\sigma} \frac{\partial \Psi_{\sigma}}{\partial x}(\vec{r})$$

Example —
Tracking through quadrupole field map in a FODO channel

- Quadrupole potentials represented with gridded transverse and longitudinal data
- Each longitudinal slice has a 2-D cubic spline interpolation model
- Space between slices linearly interpolated between spline models
- Model kicks are exact derivatives of a scalar function

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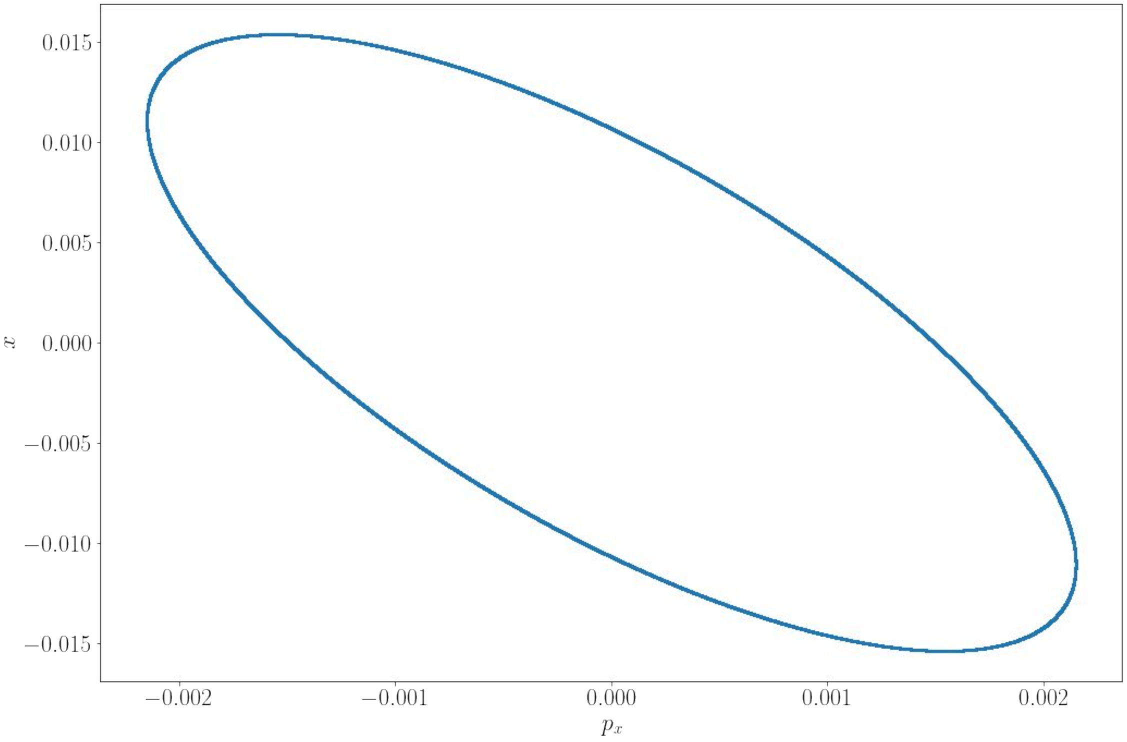

To show stability, track through a field map with perfect linear forces for 1M turns through the FODO lattice.

A non-symplectic integration scheme would show unphysical spiraling in the phase space, which we do not observe. The integrator is preserving the emittance.

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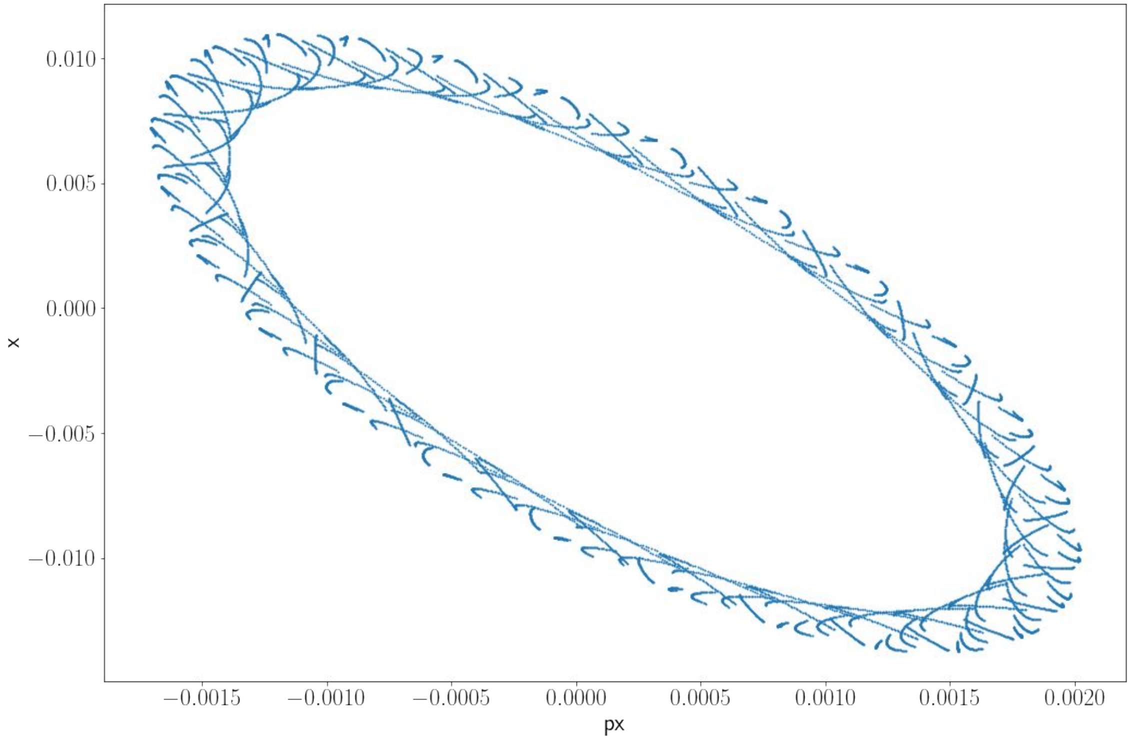

Applied 10% random noise to the potential field map, same interpolation scheme. This introduces unphysical high-order nonlinearities.

Symplectic integration captures Hamiltonian structures with nonlinear dynamics without spiraling.

Conclusions —

Use a differentiable representation for symplectic tracking

Future Work —

- Application to real fringe-fields — nontrivial 3D vector potentials
- Application to time varying systems — rf cavity mode field maps
- Self-consistent modeling — rf cavity beam loading