Diffusive models for nonlinear beam dynamics

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Introduction

In this poster

- We review a diffusive framework for describing the non-linear betatron motion by means of a Fokker-Planck equation with action-only dependent diffusion coefficient $D(I)$;
- We show how from such framework we can formulate an analytical estimation for the beam loss current;
- We present a fitting procedure for outgoing beam current and we test it on a toy model on 2 experiment-inspired scenarios.

Context of this research

- Many studies highlighted how Dynamic Aperture can be described as a function of the number of turns, in a form related to Nekhoroshev Theorem (Bazzani et al., *"Advances on the modeling of the time evolution of dynamic aperture of hadron circular accelerators"*);

$$\text{Model } 2 \Rightarrow DA(N) = \rho_* \left(\frac{\kappa}{2\mathrm{e}}\right)^\kappa \frac{1}{\ln^\kappa \frac{N}{N_0}} \tag{1}$$

- Recent studies are exploring the possibility of describing the beam distribution in terms of a diffusive model, also related to Nekhoroshev Theorem (Bazzani et al., "Analysis of the non-linear beam dynamics at top energy for the CERN large hadron collider by means of a diffusion model");
- Experimental measurements of LHC halo dynamics at different positions are available thanks to the moving collimator system (Gorzawski et al., *"Probing LHC halo dynamics using collimator loss rates at 6.5 TeV"*).

The diffusive framework in a nutshell (1/2)

We describe the betatron motion in terms of a stochastic perturbed Hamiltonian system

$$H(\theta, I, t) = H_0(I) + \varepsilon\xi(t)H_1(\theta, I) \tag{2}$$

- $H_0(I)$ regular, deterministic part of the magnetic lattice;
- $\varepsilon H_1(\theta, I)$ non-integrable part that causes the phase-space inhomogeneities, linked with Nekhoroshev estimate (ε is small);
- $\xi(t)$ stochastic noise with zero mean and unit variance.

With the Averaging Principle, we can describe the evolution of a beam distribution $\rho(I, t)$ as the solution of the Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} = \frac{\varepsilon^2}{2}\frac{\partial}{\partial I}\left(D(I)\frac{\partial \rho}{\partial I}\right) \tag{3}$$

Where $D(I)$ is the angular average of the non-integrable part $\left\langle \left(\frac{\partial H_1}{\partial \theta}\right)^2 \right\rangle_\theta$, which can be estimated via optimal remainders.

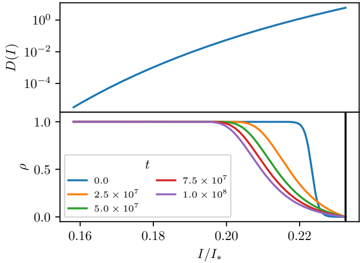
The diffusive framework in a nutshell (2/2)

The Nekhoroshev Theorem suggests the following form for the diffusion coefficient $D(I)$

$$D(I) = c \exp\left[-2\left(\frac{I_*}{I}\right)^{\frac{1}{2\kappa}}\right] \tag{4}$$

$$c^{-1} = \int_0^{I_\mathrm{a}} \exp\left[-2\left(\frac{I_*}{I}\right)^{\frac{1}{2\kappa}}\right] \mathrm{d}I \tag{5}$$

where I_a is the position of the absorbing boundary condition corresponding to the fast dynamic aperture.

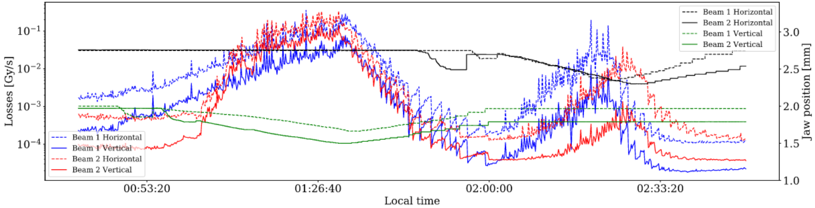


An example of a Fokker-Planck process (3) with $D(I)$ given by (4), with $\kappa = 0.33$, $I_* = 21.5$, $\varepsilon^2/2 = c$. (Top) $D(I)$. (Bottom) Crank-Nicolson integration of the process.

What is the purpose of such framework?

The goal of this line of research is to develop a valid fitting procedure for extracting the model parameters of $D(I)$ from the collimator loss rates of accelerators like LHC.

Starting from the established local diffusion measurements, we look forward to gain insight into the global beam behaviour.



[From Gorzawski et al.] Beam losses in LHC from the IC-BLM monitor. The losses are measured via moving jaws that perform the scraping on the vertical and horizontal plane separately. These measurements have been used to probe the local diffusive regime of beam halos.

Current Interpolation

It is possible to obtain a good approximation for the outgoing current at an absorbing boundary condition I_a for an initial condition $\delta(I - I_0)$

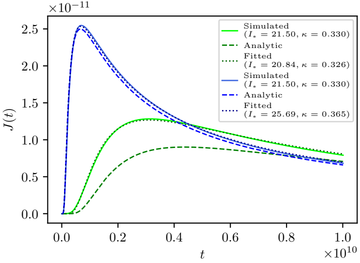
$$\mathcal{J}(t) = -\exp\left[-\frac{\left(J(I_0) + \frac{\nu(I_0)}{2}t\right)^2}{2t}\right]\frac{J(I_0)}{\sqrt{2\pi}t^{3/2}} \tag{6}$$

where we perform the change of variables

$$J(I_0) = -\int_{I_0}^{I_a} \exp\left[\left(\frac{I_*}{I}\right)^{\frac{1}{2\kappa}}\right] dI \tag{7}$$

and we linearize (4) at the initial condition I_0 for obtaining the drift term

$$\nu(I_0) = c^{1/2}\frac{\frac{1}{2\kappa}}{I_0}\left(\frac{I_*}{I_0}\right)^{\frac{1}{2\kappa}}\exp\left[\left(-\frac{I_*}{I_0}\right)^{\frac{1}{2\kappa}}\right] \tag{8}$$



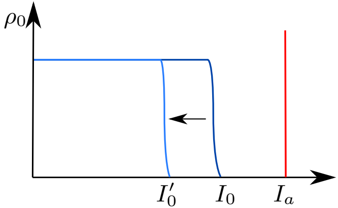
Outgoing current of a Fokker-Planck process (continuous lines), compared with the analytical approximation (dashed lines); dotted lines are a fitting procedure of κ and I_* based on the analytical approximation.

Towards a fitting procedure

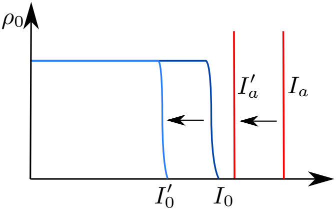
Starting from the outgoing current approximation (6) with the linearization (8), we developed some fitting procedures for reconstructing the I_* and κ values from outgoing current data.

We measured the reconstruction performance on toy models with known diffusive characters, in order to determine what are the strongest observables to use for a fitting procedure in more realistic scenarios.

We consider the following 2 scenarios:



i) Fixed absorbing boundary condition, different cutting points in the initial distribution.

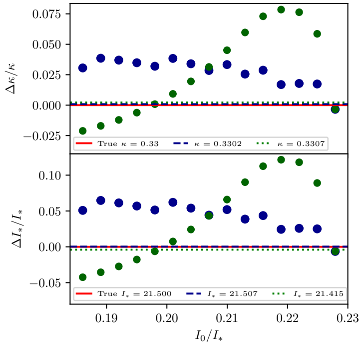


ii) Constant distance between the cut in the initial distribution and the absorbing boundary condition.

Numerical results

For both scenario i) and ii), we consider the reconstruction performance of I_* and κ, fitting either a single current profile or the timing of multiple current peaks corresponding to different values of I_0/I_*. We observe the following:

- Good reconstruction performances when $I_a - I_0$ is small;
- Transient effects depending on $I_a - I_0$ when considering only one simulation;
- Strong fitting correlation between I_* and κ
- Excellent reconstructing performances when interpolating the data from multiple simulations;



Fitting results based on Eq (6), here we try to reconstruct κ, I_* for different initial conditions and different absorbing boundary positions.

Conclusions and future work

Conclusions:

- We have shown that, within a diffusive framework, it is possible to formulate reasonable analytical estimations for the current lost.
- We have shown that these analytical estimations can be used to formulate fitting procedures for reconstructing the Nekhoroshev's terms I_* and κ.

Future work:

- Further studies are on-going to make this approach more realistic and to apply it to the collimator scans that are used to probe the beam-halo dynamics in the LHC.