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Analysis of the Chromatic Vertical Focusing Effect of Dipole Fringe Fields

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Abstract

There have been questions regarding the impact of the dipole fringe-field models (used by accelerator codes including ELEGANT and MADX) on vertical chromaticity. Here, we analyze the cause of the disagreement among codes and suggest a correction.

Introduction

The implemented models of dipole fringe field thin map in popular accelerator design codes appear in the following papers.

[1] Karl Brown, SLAC-75, 1982
[2] Etienne Forest et at, "*Second Order Fringe in MAD-X for the Module PTC*", 2006
[3] Kilean Hwang, S. Y. Lee, "*Dipole fringe field thin map for compact synchrotrons*", 2015

The linear map is known to agree with each other while there are some discrepancies on dispersion and nonlinear map. Among them, we would like to analyze the chromatic vertical focusing term that is responsible for vertical chromaticity.

Notation

Different papers use different notations. For comparison, we use the following

- edge angle: θ
- bending radius: ρ
- field integration parameter:

$$K = \int_{-\infty}^{\infty} \frac{B_y(z)\,(B_0 - B_y(z))}{gB_0} dz$$

where z is the longitudinal coordinate of the Cartesian frame perpendicular to the dipole edge so that By(z) is function of z

- vertical gap: g
- canonical momentum: $p_x = \frac{x'}{1+\delta}$ $p_y = \frac{y'}{1+\delta}$
- vertical focusing phase correction:

$$\psi = \frac{g}{\rho} K \frac{1+\sin^2\theta}{\cos\theta}$$

which is order of $O\left(g/\rho\right)$

Overview

- Review: [1] Karl Brown, SLAC-75, 1982
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Review on [1] Karl Brown, SLAC-75, 1982

The Ref. [1] is (arguably) the most widely adopted in various accelerator codes including ELEGANT and MAD. It presents the second-order Taylor map modeling the fringe field effect in terms of the matrix and the tensor elements. Among them, we would like to focus on the R_{43} and T_{436} elements that are responsible for the on/off-momentum vertical focusing effect. It is also important to mention that the matrix and the tensor acts on the following 6D phase-space variables

$$(x, x', y, y', t, \delta)$$

where t is proportional to the transient time deviation, prime indicate differential along the reference orbit, and the last element δ is fractional momentum deviation.

Entrance

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= 0 \\
T_{111} &= -(h/2)\tan^2\beta_1 \\
T_{133} &= (h/2)\sec^2\beta_1 \\
R_{21} &= -(1/f_x) = h\tan\beta_1 \\
R_{22} &= 1 \\
T_{211} &= (h/2R_1)\sec^3\beta_1 - nh^2\tan\beta_1 \\
T_{212} &= h\tan^2\beta_1 \\
T_{216} &= -h\tan\beta_1 \\
T_{233} &= h^2(n + \tfrac{1}{2} + \tan^2\beta_1)\tan\beta_1 - (h/2R_1)\sec^3\beta_1 \\
T_{234} &= -h\tan^2\beta_1 \\
R_{33} &= 1 \\
R_{34} &= 0 \\
T_{313} &= h\tan^2\beta_1 \\
&\boxed{R_{43} = -(1/f_y) = -h\tan(\beta_1 - \psi_1)} \\
R_{44} &= 1 \\
T_{413} &= -(h/R_1)\sec^3\beta_1 + 2h^2 n\tan\beta_1 \\
T_{414} &= -h\tan^2\beta_1 \\
T_{423} &= -h\sec^2\beta_1 \\
&\boxed{T_{436} = h\tan\beta_1 - h\psi_1\sec^2(\beta_1 - \psi_1)}
\end{aligned}
$$

Exit

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= 0 \\
T_{111} &= (h/2)\tan^2\beta_2 \\
T_{133} &= -(h/2)\sec^2\beta_2 \\
R_{21} &= -1/f_x = h\tan\beta_2 \\
R_{22} &= 1 \\
T_{211} &= (h/2R_2)\sec^3\beta_2 - h^2(n + \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2 \\
T_{212} &= -h\tan^2\beta_2 \\
T_{216} &= -h\tan\beta_2 \\
T_{233} &= h^2(n - \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2 - (h/2R_2\sec^3\beta_2 \\
T_{234} &= h\tan^2\beta_2 \\
R_{33} &= 1 \\
R_{34} &= 0 \\
T_{313} &= -h\tan^2\beta_2 \\
&\boxed{R_{43} = -1/f_y = -h\tan(\beta_2 - \psi_2)} \\
R_{44} &= 1 \\
T_{413} &= -(h/R_2)\sec^3\beta_2 + h^2(2n + \sec^2\beta_2)\tan\beta_2 \\
T_{414} &= h\tan^2\beta_2 \\
T_{423} &= h\sec^2\beta_2 \\
&\boxed{T_{436} = h\tan\beta_2 - h\psi_2\sec^2(\beta_2 - \psi_2)}
\end{aligned}
$$

Review on [1] Karl Brown, SLAC-75, 1982

Using our notation

to the first order of ψ :

$$R_{43} = T_{436} = -\frac{\tan\theta}{\rho} - \frac{\psi}{\rho}\sec^2\theta + O\left(\psi^2\right)$$

Recall that, the matrix and the tensor act on the phase-space, resulting:

$$\Delta y' = \left(-\frac{\tan\theta}{\rho} - \frac{\psi}{\rho}\sec^2\theta\right)(1+\delta)\, y$$

However, in terms of the canonical momentum the vertical focusing is <u>independent of the energy offset</u>

$$\Delta p_y = \Delta\frac{y'}{1+\delta} = \left(-\frac{\tan\theta}{\rho} - \frac{\psi}{\rho}\sec^2\theta\right) y$$

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Review on
[2] Etienne Forest et at, *"Second Order Fringe in MAD-X for the Module PTC"*, 2006

The Ref. [2] is adopted in the MAD-X PTC module. It presents the generating function-based map to the 2nd order of field strength. Equation (32) of the Ref. [2] shows the 2nd order of field effect on vertical momentum:

$$\begin{aligned} d\Delta p_y^{11}/dy\big|_{y=0} &= g b_0^2 K \left\{ [x, x'] + x'^2\, [y, y'] \right\} \\ &= g b_0^2 K \left\{ \underbrace{\frac{(1+\delta)^2 - p_y^2}{p_z^3}}_{[x,x']} + \underbrace{\frac{p_x^2}{p_z^2}}_{x'^2} \underbrace{\frac{(1+\delta)^2 - p_x^2}{p_z^3}}_{[y,y']} \right\} \end{aligned} \tag{32}$$

Using our notation

Plugging the followings into the Eq. (32) of Ref. [2]

$$p_x = (1+\delta)\sin\theta \qquad\qquad p_z = \sqrt{(1+\delta)^2 - p_x^2}$$

The vertical focusing (2nd order of the field strength) reads

$$\Delta p_y^{11} = -\frac{\psi}{\rho}\sec^2\theta y + \frac{\psi}{\rho}\sec^2\theta y\delta$$

Note that the chromatic vertical focusing is the first order of ψ and equal to the negative of the first order of ψ term of the on-momentum vertical focusing.

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Review on
[3] Kilean Hwang, S. Y. Lee, "*Dipole fringe field thin map for compact synchrotrons*", 2015

The Ref. [3] is adopted in ELEGANT. It presents the single exponent Lie map (to the 4th order of canonical variables) and corresponding Taylor map (to the 3rd order of canonical variables). Recently, ELEGANT manipulated the Lie map to implement explicit symplectic map. The Eq. (43) and (47) of Ref. [3] show the vertical Taylor map. Taking the vertical focusing term only,

$$\Delta p_z = -z\left[\frac{\tan\theta_E}{\rho} - \frac{1+\sin^2\theta_E}{(1+\delta)\cos^3\theta_E}\frac{g}{\rho^2}K_2\right]$$

where z is the vertical coordinate following S.Y.Lee's book

Using our notation

Expanding to the first order of δ , the vertical focusing reads

$$\Delta p_y = \left(-\frac{\tan\theta}{\rho} - \frac{\psi}{\rho}\sec^2\theta \right) y + \frac{\psi}{\rho}\sec^2\theta y\delta$$

Note that the chromatic vertical focusing agrees with the Ref. [2]

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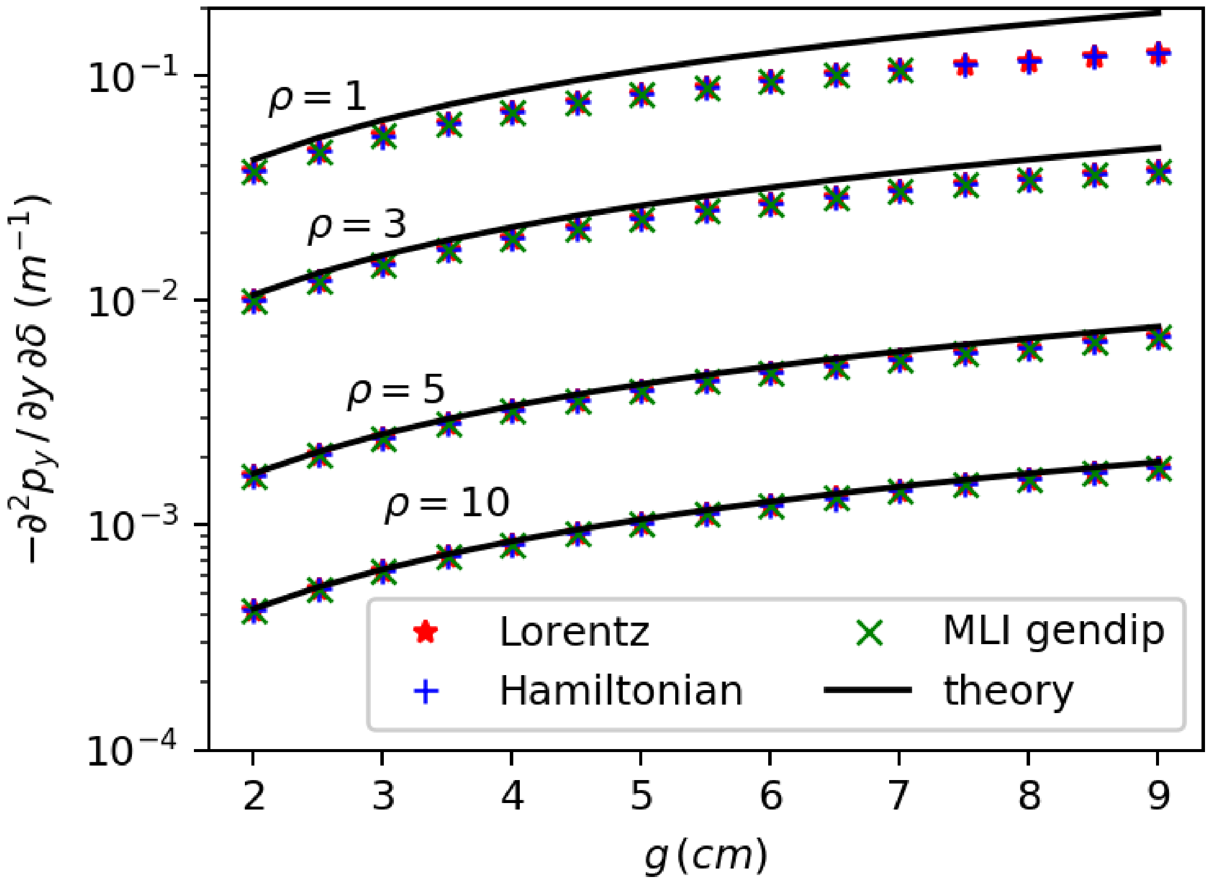
Simulation Tools

In order to verify the chromatic vertical focusing term, we compare using various simulation tools

- Lorentz force tracking with an automatic differentiation library.
- Hamiltonian (Eq. (16) of Ref. [3]) tracking with an automatic differentiation library.
- MaryLie/Impact code that use Lie map tracking and is capable of producing Taylor coefficients.

Taylor coefficient of chromatic vertical focusing term

We model fringe field strength using sigmoid function. We also set the edge angle by $\pi/4$. The label 'theory' corresponds to the Ref. [2] and the Ref. [3]. Recall that the chromatic vertical focusing on canonical momentum is zero in the Ref [1].



IOTA vertical chromaticity

IOTA is a storage ring built as a test bed for novel accelerator physics technologies. We took a version of the IOTA design lattice that is for the nonlinear integrable optics test. Recall that as the bending radii become smaller the fringe field effects become more important. IOTA ring has two types of sector dipoles whose bending radii are `0.822, 0.773(m)`. Assuming 2.5 cm vertical half gap, and the field integration parameter K=0.5, the optics parameter using Ref. [1] and [3] model of the dipole fringe field are:

	ν_x	$\partial\nu_x/\partial\delta$	ν_y	$\partial\nu_y/\partial\delta$
Ref. [1]	0.3	4.072	0.3	0.487
Ref. [2,3]	0.3	4.072	0.3	0.581

Note that relative difference in the vertical chromaticity may not neglectable.

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Conclusions and Remarks

- We compared the chromatic vertical focusing of dipole fringe field thin map in three papers that are often adopted in accelerator simulation codes.

$$\frac{\partial^2 \Delta p_y}{\partial y \partial \delta} = \begin{cases} 0 & \text{Ref. [1]} \\ \frac{\psi}{\rho} \sec^2 \theta & \text{Ref. [2, 3]} \end{cases}$$

- We saw that for a compact ring like the IOTA, dipole fringe field contribution on vertical chromaticity may not ignorable.
- It is also worth mentioning that the dispersion error due to the dipole fringe field is often ignored in most of the accelerator simulations. However, it can be more important than the chromatic vertical focusing effect as it introduces the energy-dependent closed orbit deviation. Ref. [3] indicate that the chromatic closed orbit deviation is order of

$$x_{closed} \sim O(\delta g^2/\rho)$$