ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-WEPAB248

KURTH VLASOV-POISSON SOLUTION FOR A BEAM IN THE PRESENCE OF TIME-DEPENDENT ISOTROPIC FOCUSING*

C. E. Mitchell†, K. Hwang, R. D. Ryne
Lawrence Berkeley National Laboratory, Berkeley, CA, USA

Abstract

The well-known K-V distribution provides an exact solution of the self-consistent Vlasov-Poisson system describing an unbunched charged particle beam with nonzero temperature in the presence of time-dependent linear transverse focusing. We describe a lesser-known exact solution of the Vlasov-Poisson system that is based on the work of Kurth in stellar dynamics. Unlike the K-V distribution, the Kurth distribution is a true function of the phase space variables, and the solution may be constructed on either the 4D or 6D phase space, for the special case of isotropic linear focusing. Numerical studies are performed for benchmarking simulation codes, and the stability properties of a 4D Kurth distribution are compared with those of a K-V distribution.

INTRODUCTION

The K-V distribution [1] has played a central role in understanding the self-consistent dynamics of beams with space charge in linacs and storage rings. However, the K-V distribution is not a true function (but a measure concentrated on an ellipsoidal hypershell), and it is sensitive to a range of well-studied collective instabilities. In addition, the K-V construction of producing linear forces by using a delta function in a single invariant of motion cannot be generalized from 4D to 6D [2].

Exact solutions of the (gravitational) Vlasov-Poisson system are well-known in stellar dynamics. These solutions are generally 6D and invariant under rotations of the spatial and momentum variables (isotropic), providing additional invariants of motion. One such exact solution for a galaxy confined by linear self-forces developed by Kurth [3] allows for both steady-state and oscillating solutions. It appears to be widely used in the stellar dynamics community [4].

With minor changes to allow for time-dependent linear external focusing, the Kurth solution can be adapted to treat the case of charged-particle beams in either 4D or 6D. Indeed, this distribution may be interpreted as a special case of [5]. The Kurth distribution can be a valuable tool for benchmarking treatments of time-dependent 2D or 3D space charge. We summarize the theory of this distribution, and illustrate numerical studies comparing its stability properties with those of a K-V beam.

* Work supported by DOE Office of Science, Office of High Energy Physics
† chadmitchell@lbl.gov

KURTH DISTRIBUTION

Unbunched Beam (4D)

The Hamiltonian for a long coasting beam with axially-symmetric linear external focusing, with s as the independent variable, and with momenta normalized to the design momentum $p_0 = mc\beta\gamma$ is given by:

$$H(x,p_x,y,p_y,s) = \frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}\Omega(s)^2(x^2+y^2)+\Phi. \quad (1)$$

Here Φ denotes the 2D space charge potential:

$$\Phi(x,y,s) = \frac{q\phi(x,y,s)}{\beta^2\gamma^3 mc^2}, \qquad \nabla_\perp^2\phi = -\frac{\rho}{\epsilon_0}. \quad (2)$$

Let R be any solution of the 2D envelope equation:

$$R''(s) + \Omega(s)^2 R(s) - \frac{K_{pv}}{R(s)} - \frac{\epsilon^2}{R(s)^3} = 0, \quad (3)$$

where K_{pv} is the generalized perveance and $\epsilon > 0$ denotes the beam edge emittance. In terms of the dimensionless coordinates:

$$x_N = x/R(s), \qquad p_{xN} = [R(s)p_x - R'(s)x]/\epsilon, \quad (4a)$$

$$y_N = y/R(s), \qquad p_{yN} = [R(s)p_y - R'(s)y]/\epsilon, \quad (4b)$$

define the quantities:

$$E = \frac{1}{2}(p_{xN}^2 + p_{yN}^2 + x_N^2 + y_N^2), \quad (5a)$$

$$L_z = x_N p_{yN} - y_N p_{xN}. \quad (5b)$$

The Kurth distribution then takes the form:

$$f(x,p_x,y,p_y,s) = \frac{1}{2\pi^2\epsilon^2}\left(1-2E+L_z^2\right)_+^{-1/2}, \quad |L_z|<1 \quad (6)$$

where f is taken to vanish if the quantity in brackets is negative or if $|L_z| \geq 1$. After integrating over momenta, one finds that the 2D spatial density is uniform within a disk of radius R, so that the space charge fields are linear, and the rms emittance is given by:

$$\epsilon_{x,rms} = \epsilon_{y,rms} = \frac{\epsilon}{4}. \quad (7)$$

Although the distribution (6) is expressed using L_z, f in fact has zero mean angular momentum, $\langle L_z\rangle = 0$.

One may verify using (1) and (3) that the quantities (5) are invariants of the single-particle motion. It follows that (6) is a self-consistent solution of the Vlasov-Poisson system:

$$\frac{\partial f}{\partial s} + \{f,H\} = 0, \quad \nabla^2\Phi = -2\pi K_{pv}\int f dp_x dp_y.$$

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-WEPAB248

The distribution (6) may be compared with that of a K-V beam with the same second moments:

$$f_{KV}(x,p_x,y,p_y,s) = \frac{1}{\pi^2\epsilon^2}\delta(1-2E). \quad (8)$$

Like the K-V beam, the 2D projections $X-Y$, P_x-P_y, $X-P_x$, and $Y-P_y$ of (6) are uniformly-filled ellipses. However, the coordinates X, P_y are statistically independent, as are Y, P_x.

Bunched Beam (6D)

The Hamiltonian for a bunched beam with spherically-symmetric linear external focusing, with $\tau = ct$ as the independent variable, and with momenta normalized by mc is given by:

$$H(x,p_x,y,p_y,z,p_z,\tau) = \frac{1}{2}|\mathbf{p}|^2 + \frac{1}{2}\Omega(\tau)^2|\mathbf{r}|^2 + \Phi, \quad (9)$$

where $\mathbf{r} = (x,y,z)$, $\mathbf{p} = (p_x,p_y,p_z)$, and Φ denotes the 3D space charge potential:

$$\Phi(x,y,z,\tau) = \frac{q\phi(x,y,z,\tau)}{mc^2}, \quad \nabla^2\phi = -\frac{\rho}{\epsilon_0}. \quad (10)$$

Let R be any solution of the 3D envelope equation:

$$R''(\tau) + \Omega(\tau)^2R(\tau) - \frac{r_cN}{R(\tau)^2} - \frac{\epsilon^2}{R(\tau)^3} = 0, \quad (11)$$

where r_c denotes the classical particle radius, N denotes the bunch population, and $\epsilon > 0$ is a 3D edge emittance. In terms of dimensionless coordinates:

$$\mathbf{r}_N = \mathbf{r}/R(s), \qquad \mathbf{p}_N = [R(s)\mathbf{p} - R'(s)\mathbf{r}]/\epsilon, \quad (12)$$

define the quantities:

$$E = \frac{1}{2}(|\mathbf{p}_N|^2 + |\mathbf{r}_N|^2), \qquad L = |\mathbf{r}_N \times \mathbf{p}_N|. \quad (13)$$

The Kurth distribution is given by:

$$f(x,p_x,y,p_y,z,p_z,\tau) = \frac{3}{4\pi^3\epsilon^3}\left(1-2E+L^2\right)_+^{-1/2}, \quad L<1 \quad (14)$$

After integrating over momenta, one finds that the 3D spatial density is uniform within a ball of radius R, so that the space charge fields are linear, and the rms emittance is given by:

$$\epsilon_{x,rms} = \epsilon_{y,rms} = \epsilon_{z,rms} = \frac{\epsilon}{5}. \quad (15)$$

Although the distribution (14) is expressed using L, f in fact has zero mean angular momentum about each axis.

One may verify using (9) and (11) that the quantities (13) are invariants of the single-particle motion. It follows that (14) is a self-consistent solution of the Vlasov-Poisson system:

$$\frac{\partial f}{\partial \tau} + \{f,H\} = 0, \quad \nabla^2\Phi = -4\pi r_cN\int f dp_xdp_ydp_z.$$

Although Kurth's original paper focused on the gravitational 6D case (without external focusing), the numerical studies below focus on the 4D case.

NUMERICAL STUDIES

Consider a 200 MeV unbunched (4D) proton beam with an edge emittance of $\epsilon = 10.14$ mm-mrad and a current of $I = 20$ A. Simulations in a linear focusing channel were performed using a 1D Poisson solver based on Gauss' law and particle sorting (assuming symmetry about the axis of motion), where 100K particles were used. Similar results were obtained using a 2D spectral Poisson solver.

Breathing Beam in a Constant Focusing Channel

External focusing (corresponding to $B = 2.7$ T) was chosen to produce a space charge tune depression ratio of $\omega/\omega_0 = 0.74$. To introduce nontrivial s-dependence, the initial beam size was mismatched by 10%. (See Fig. 1.) The uniformity of the spatial distribution of the Kurth beam is well-preserved over 450 envelope periods.

Since both beams experience linear space charge fields, the beam emittance should be well-preserved. Figure 2 shows the relative emittance growth for both a Kurth and a K-V distribution. While the emittance fluctuations in the Kurth beam are consistent with those expected due to numerical particle noise, the K-V beam shows the onset of an instability after 60 envelope periods, and the beam develops radial density waves. This indicates that the two distributions can exhibit quite different stability behavior.

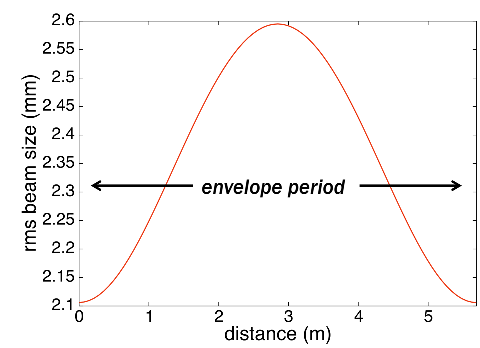

Figure 1: One envelope period for a beam in a constant-focusing channel with $\omega/\omega_0 = 0.74$ and 10% mismatch.

Matched Beam in a Periodic Focusing Channel

To study a system with periodic focusing, we considered a single period consisting of a drift and a constant focusing section of equal length. The focusing strength was chosen to provide an undepressed and a depressed phase advance per period of $\sigma_0 = 84.4°$ and $\sigma = 47.6°$, respectively, yielding a tune depression ratio $\sigma/\sigma_0 = 0.56$. (See Fig. 3.)

Matched Kurth and K-V beams were tracked for 450 lattice periods, and both beams show relative fluctuations in ϵ_x and ϵ_y of $< 4\times10^{-4}$. This stability is reflected in the preservation of the uniform spatial density, shown in the linear behavior of the radial profile in Fig. 4 for the Kurth beam.

However, the Kurth beam is not always stable. As a second example, we chose the focusing strength to provide an undepressed and a depressed phase advance per period of

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-WEPAB248

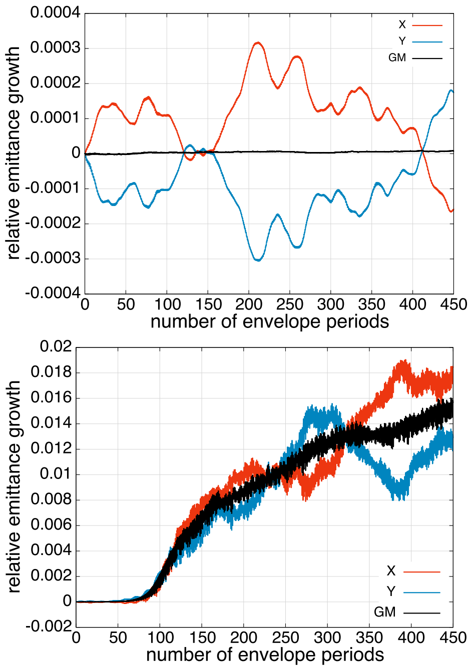

Figure 2: Relative emittance growth in the channel of Fig. 1, corresponding to ϵ_x, ϵ_y, and $\sqrt{\epsilon_x \epsilon_y}$. (Upper) For a Kurth distribution. (Lower) For a K-V distribution.

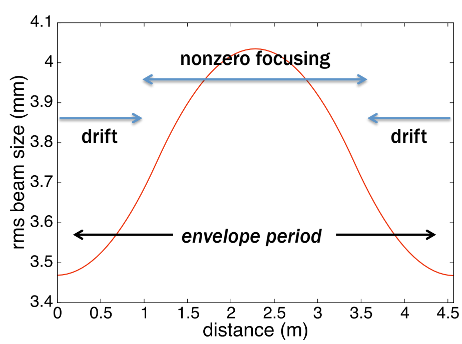

Figure 3: One envelope period for a beam in a periodic channel with $\sigma_0 = 84.4°$ and $\sigma = 47.6°$.

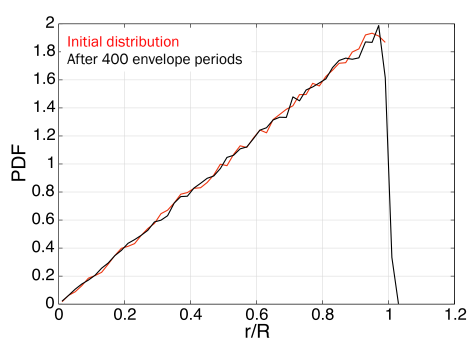

Figure 4: Radial profile for the Kurth beam before and after 400 periods of tracking, showing that the uniform spatial density is well-preserved.

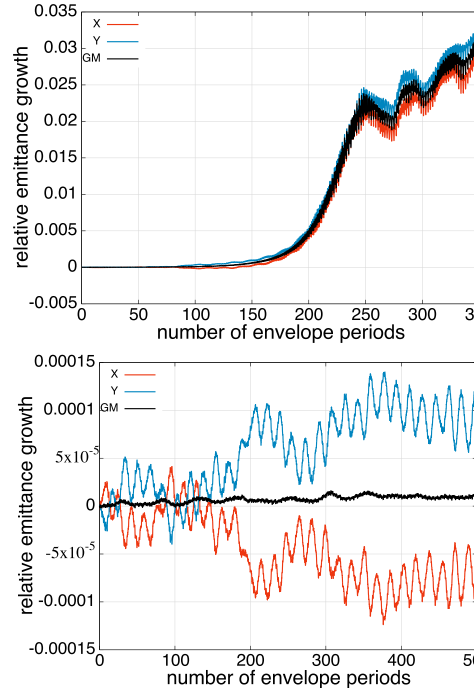

Figure 5: Relative emittance growth for a Kurth distribution (upper) and a K-V distribution (lower) in a periodic channel with $\sigma_0 = 125°$ and $\sigma = 83.8°$.

$\sigma_0 = 125°$ and $\sigma = 83.8°$, respectively, yielding a tune depression ratio $\sigma/\sigma_0 = 0.67$. The Kurth beam now shows the onset of an instability after 140 lattice periods (Fig. 5). In this case, a similar instability does not appear in the K-V beam.

CONCLUSION

The Kurth distribution is a simple self-consistent solution of the Vlasov-Poisson system describing an intense beam in the presence of linear, isotropic *s*-dependent focusing. Essentially the same functional form for the distribution is used in both 4D and 6D. From Eqs. (3) and (11), one may verify that solutions satisfy the usual rms envelope equations [6]. Numerical simulations were used to verify the preservation of a Kurth distribution during self-consistent tracking with space charge. The distribution exhibits instability behavior distinct from that of a corresponding K-V beam. Future work is needed to examine the stability properties of the Kurth distribution in detail.

ACKNOWLEDGMENTS

This work was was supported by the Director, Office of Science, Office of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and made use of computer resources at the National Energy Research Scientific Computing Center. The authors acknowledge support from the U.S. DOE Early Career Research Program under the Office of High Energy Physics.

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-WEPAB248

REFERENCES

[1] I. M. Kapchinskij and V. V. Vladimirskij, "Limitations of Proton Beam Current in a Strong Focusing Linear Accelerator Associated with the Beam Space Charge", in *Proc. of the Int. Conf. on High Energy Accelerators and Instrumentation*, CERN, Geneva, Switzerland, 1959, pp. 274-288.

[2] F. Sacherer, "Transverse Space-Charge Effects in Circular Accelerators", Appendix A, Ph.D. dissertation, University California, Berkeley, 1968.

[3] R. Kurth, "A Global Particular Solution to the Initial-Value Problem of Stellar Dynamics", *Quarterly of Applied Mathematics*, vol. 36, no. 3, pp. 325-329, 1978. doi:10.1090/qam/508777

[4] T. Ramming and G. Rein, "Oscillating Solutions of the Vlasov-Poisson System–a Numerical Investigation", *Physica D*, vol. 365, pp. 72-79, 2018. doi:10.1016/j.physd.2017.10.013

[5] V. Danilov *et al*, "Self-Consistent Time Dependent Two Dimensional and Three Dimensional Space Charge Distributions with Linear Force", *Phys. Rev. ST. Accel. Beams*, vol. 6, p. 094202, 2003. doi:10.1103/PhysRevSTAB.6.094202

[6] F. Sacherer, "RMS Envelope Equations with Space Charge", *IEEE Transactions on Nuclear Science*, vol. 18, no. 3, p. 1105, 1971. doi:10.1109/TNS.1971.4326293