

INVESTIGATION OF VLASOV SYSTEMS WITH A CERTAIN CLASS OF LINEARLY-COLLECTIVE HAMILTONIANS

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Abstract

In many cases the Vlasov equation cannot be solved exactly due its inherent non-linearity arising from collective terms in the Hamiltonian. Based on the analysis of the Hamiltonian's dependence on the phase-space density and the requirement for self-consistency in this contribution a class of Hamiltonians is defined and characterized. For members of this class the corresponding expansion of the Vlasov equation terminates. The new, potentially non-autonomous, Hamiltonian of the resulting Liouville equation depends only on the initial condition of the phase-space density. Prominent members of this class are Poisson-type kick-Hamiltonians, which we show as an example. We expect these investigations to be a potential starting point for the analysis and conception of operator-splitting schemes or splitting-free methods for beam-dynamics simulation codes.

INTRODUCTION

The Liouville equation

$$\begin{cases} \frac{\partial \Psi}{\partial t} - \{H, \Psi\} = 0 \\ \Psi(0, z) = \Psi_0(z) \end{cases} \quad (1)$$

is the evolution equation for the phase-space density (PSD) $\Psi(t, z) : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ of a system of non-interacting, identically independently distributed (iid) particles with the Hamiltonian $H(t, z) : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, where n is the number of *geometric* degrees of freedom, $\Psi(t, z) dz$ is the probability that *any* of the particles is in the PS volume element dz at time t , and $\{\cdot, \cdot\}$ denotes the Poisson bracket

$$\{u, v\} \equiv (\nabla_z u)^T J \nabla_z v \equiv \sum_{i,j=0}^{2n} J_{ij} \frac{\partial u}{\partial z_i} \frac{\partial v}{\partial z_j}. \quad (2)$$

$J \in \mathbb{R}^{2n \times 2n}$ is an antisymmetric matrix, whose actual representation depends on the choice of the base of the phase space. The Liouville equation therefore defines a linear first-order differential equation, that can be readily solved using the method of characteristics [1]. Given the (generally non-linear) symplectic (i.p. injective) flow $\phi(t_1, t_0, z) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ of the Hamiltonian, an explicit solution to the initial value problem (IVP) (1) is given by

$$\Psi(t, z) = \Psi_0(\phi(0, t, z)), \quad (3)$$

where have used $\phi^{-1}(t, 0, \cdot) = \phi(0, t, \cdot)$. Hence, solving Liouville's equation is as hard as solving Hamilton's equation of motion.

The Vlasov equation

$$\begin{cases} \frac{\partial \Psi}{\partial t} - \{H[\Psi], \Psi\} = 0 \\ \Psi(0, z) = \Psi_0(z) \end{cases} \quad (4)$$

extends the Liouville equation to include interactions of iid particles in the mean field approximation. Here this mean field interaction is accounted for by introducing a dependence of the Hamiltonian $H[\Psi]$ itself on the PSD. Due to this dependence Eq. (4) is a non-linear partial differential equation. Consequently, the method of characteristics is not directly applicable anymore and exact solutions can generally not be found. Nevertheless, the method of characteristics has been successfully used in the field of collective beam dynamics (within which it is also known as the Perron-Frobenius method) to construct approximate solutions of the Vlasov equation [2, 3].

In this contribution we show that for certain Hamiltonians exact solutions of Eq. (4) indeed can be found. To this end we define a class of Hamiltonians, for which the Vlasov Equation degenerates in to an effective Liouville equation, making it explicitly solvable.

EXPANDING THE VLASOV EQUATION

A formal solution to the IVP (4) can be obtained by integrating over t

$$\Psi(t, z) = \Psi_0(z) + \int_0^t \frac{\partial \Psi}{\partial t'} dt' \quad (5)$$

$$= \Psi_0(z) + \int_0^t \{H[\Psi], \Psi\}|_{t'} dt' \quad (6)$$

Putting this expression for Ψ back into the Vlasov Equation (4) yields

$$\frac{\partial \Psi}{\partial t} - \left\{ H \left[\Psi_0 + \int_0^t \{H[\Psi], \Psi\}|_{t'} dt' \right], \Psi \right\} = 0, \quad (7)$$

where we have substituted Ψ only in the first argument of the Poisson bracket. We note that the non-linearity of the resulting expanded Vlasov equation is now isolated in the time-integral term.

This substitution can be repeated iteratively for Eq. (7) and the resulting equations, leading to an iteration procedure closely related to the concepts of Magnus expansion and Picard iteration. If at each iteration step the substitution is carried out for all occurrences of Ψ on the left-hand side of the outermost Poisson bracket only (leaving the Ψ on the right-hand side untouched) the basic structure of an Vlasov equation retained. One might be hopeful that the Liouville equation obtained by truncating that iteration at some order and neglecting any further occurrences of Ψ yields a

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good approximation of the original Vlasov equation. In this contribution, however, we will focus on a class of collective Hamiltonians for which this expansion truncates exactly after the first iteration.

AFFINE LINEARLY COLLECTIVE HAMILTONIANS

Assume now that $H[\Psi]$ is not explicitly time dependent and is affine linear in its collective dependence, in the sense that for $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} H[\alpha\Psi + \beta\Phi] &= H_c[\alpha\Psi + \beta\Phi] + H_0 & (8) \\ &= \alpha H_c[\Psi] + \beta H_c[\Phi] + H_0, & (9) \end{aligned}$$

where H_c is the collective Hamiltonian and H_0 is a single-particle Hamiltonian with no dependence on the phase-space density. Additionally exploiting the linearity of Poisson's bracket, Eq. (7) then becomes

$$\frac{\partial\Psi}{\partial t} - \{H[\Psi_0], \Psi\} - \int_0^t \{H_c[\{H[\Psi], \Psi\}_{t'}], \Psi\} dt' = 0. \quad (10)$$

Hence, if for all Ψ

$$H_c[\{H[\Psi], \Psi\}] = 0 \quad (11)$$

then the Vlasov equation for this Hamiltonian degenerates to

$$\frac{\partial\Psi}{\partial t} - \{H[\Psi_0], \Psi\} = 0, \quad (12)$$

which effectively is a Liouville Equation, as the Hamiltonian does not depend on the evolution of the PSD Ψ but only on its initial condition Ψ_0 . As Eq. (11) needs to hold for any choice of Ψ it follows that it has to hold for the collective and non-collective part of the Hamiltonian in the Poisson bracket individually

$$H_c[\{H_c[\Psi], \Psi\}] = 0 \quad (13)$$

$$\text{and } H_c[\{H_0, \Psi\}] = 0. \quad (14)$$

Equation (11) states the condition that the collective part of the Hamiltonian needs to be invariant under the time-derivate of Ψ , which in turn is given by the Poisson bracket of the Hamiltonian itself and Ψ . Hence, it can be interpreted as the defining property of a class of self-preserving Hamiltonians.

Convolution Hamiltonians

Consider the case, where the collective part of the Hamiltonian is determined by the convolution of the PSD with a function $G: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $H_c = H_G$ with

$$H_G[\Psi](t, z) = (G * \Psi)(t, z) \quad (15)$$

$$\equiv \int_{\mathbb{R}^{2n}} G(z, z') \Psi(t, z') dz'. \quad (16)$$

By construction, the collective dependence of this type of Hamiltonian is linear so that the previous considerations are applicable. Now we want to investigate for which choice of G and H_0 Eqs. (13) and (14) hold.

Plugging in H_G , the left hand side of Eq. (13) yields

$$\begin{aligned} H_G[\{H_G[\Psi], \Psi\}] &= \int_{\mathbb{R}^{2n}} G(z, z') \\ &\left\{ \int_{\mathbb{R}^{2n}} G(z', z'') \Psi(t, z'') dz'', \Psi(t, z') \right\}_{z'}, dz', \end{aligned} \quad (17)$$

where the notation $\{\cdot, \cdot\}_{z'}$ indicates that the Poisson bracket acts on the coordinates z' . Consequently the z'' -convolution, as well as $\Psi(t, z'')$ can be taken out of the Poisson bracket

$$\begin{aligned} H_G[\{H_G[\Psi], \Psi\}] &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \\ &\Psi(t, z'') G(z, z') \{G(z', z''), \Psi(t, z')\}_{z'} dz' dz''. \end{aligned} \quad (18)$$

Using an identity for expressions of the form $\int_{\mathbb{R}^{2n}} \{u, v\} w dz$, shown in the appendix, see Eq. (40), we get

$$\begin{aligned} H_G[\{H_G[\Psi], \Psi\}] &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \\ &\Psi(t, z'') \Psi(t, z') \{G(z, z'), G(z', z'')\}_{z'} dz' dz'', \end{aligned} \quad (19)$$

and therefore

$$\begin{aligned} \{G(x, z), G(z, y)\} &= 0 \quad \forall x, y \in \mathbb{R}^{2n} \\ \Rightarrow H_G[\{H_G[\Psi], \Psi\}] &= 0 \quad \forall \Psi. \end{aligned} \quad (20)$$

Turning to Eq. (14) we analogously see that

$$\begin{aligned} H_G[\{H_0, \Psi\}] &= \int_{\mathbb{R}^{2n}} G(z, z') \{H_0, \Psi(t, z')\}_{z'} dz' \end{aligned} \quad (21)$$

$$= \int_{\mathbb{R}^{2n}} \Psi(t, z') \{G(z, z'), H_0\}_{z'} dz' \quad (22)$$

so that

$$\begin{aligned} \{G(x, z), H_0\} &= 0 \quad \forall x \in \mathbb{R}^{2n} \\ \Rightarrow H_G[\{H_0, \Psi\}] &= 0 \quad \forall \Psi. \end{aligned} \quad (23)$$

Example 1: Consider $G(x, z) = f(x)f(z)$, with $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Equation (20) holds due to the anti-symmetry of the Poisson bracket

$$\{G(x, z), G(z, y)\} = f(x)f(y) \{f(z), f(z)\} = 0. \quad (24)$$

Looking at the Hamiltonian, it becomes clear why it is self-preserving

$$H_G[\Psi] = f(z) \int_{\mathbb{R}^{2n}} f(z') \Psi(t, z') dz' \quad (25)$$

$$= f(z) E_\Psi[f]. \quad (26)$$

As the expected value $E_\Psi[f]$ is merely a global factor, the PSD will evolve along the level-sets of $f(z)$, which in turn however preserves the expected value.

Concerning the non-collective part, we see from Eq. (23) that H_0 can be any function for which the Poisson bracket with f vanishes

$$\{G(x, z), H_0\} = f(x) \{f(z), H_0\}. \quad (27)$$

Example 2: Consider $G(x, z) = f(M(x), M(z))$, with $M: \mathbb{R}^{2n} \rightarrow \mathbb{R}^m$ and $f: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$, $m \in \mathbb{N}$. In this case we see that

$$\{G(x, z), G(z, y)\} = (\nabla_2 f)^T DM(z) J (DM(z))^T \nabla_1 f, \quad (28)$$

where DM is the Jacobi matrix of M and $\nabla_1 f$ and $\nabla_2 f$ denote the gradient of f with respect to its first and second argument, respectively. Hence, we see that the self-conservation condition holds for any f , if the term involving the Jacobi matrices vanishes

$$\begin{aligned} DM(z) J (DM(z))^T &= 0 \quad \forall z \in \mathbb{R}^{2n} \\ \Rightarrow H_G[\{H_G[\Psi], \Psi\}] &= 0 \quad \forall \Psi. \end{aligned} \quad (29)$$

We note that for $n = 1$, $m = 2$ this condition is equivalent to $\det(DM) = 0$.

Similarly, we see for the non-collective part

$$\{G(x, z), H_0\} = (\nabla_2 f)^T DM(z) J \nabla H_0 \quad (30)$$

so that the self-consistency condition, see Eq. (23), is fulfilled if the gradient of the non-collective Hamiltonian is locally contained in the kernel of $DM(z) J$

$$\begin{aligned} \nabla H_0 \in \ker(DM(z) J) \quad \forall z \in \mathbb{R}^{2n} \\ \Rightarrow H_G[\{H_0, \Psi\}] = 0 \quad \forall \Psi. \end{aligned} \quad (31)$$

Poisson Kick Hamiltonians

A prominent example which is relevant in the dynamics of charged particle beams – for instance in the study of microbunching gain induced by longitudinal space-charge in ultra relativistic linacs with bunch compression chicane – are Poisson-kick Hamiltonians [3–6]. Here, the collective part of the Hamiltonian depends linearly on the PSD via an interaction potential ϕ , which is a function of the canonical coordinates $q = (z_1, \dots, z_n)$ and is given as a solution of an inhomogeneous Poisson equation, where the source term is the spatial density, i.e. the projection of the PSD along the conjugate momenta $p = (z_{n+1}, \dots, z_{2n})$,

$$\nabla_q^2 \phi(q) = \int_{\mathbb{R}^n} \Psi(q, p') dp'. \quad (32)$$

The solution to Poisson's equation can be constructed as a convolution of the source term with an appropriate kernel K so that

$$H_K[\Psi](q, p) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(q, q') \Psi(q', p') dp' dq'. \quad (33)$$

By introducing the matrix $Q \in \mathbb{R}^{2n \times 2n}$ which maps a phase-space vector to the canonical variables $Q = (I_n \ 0_n)$, we can see that findings of the above example are indeed applicable to this Hamiltonian, as we can write

$$H_K[\Psi](z) = \int_{\mathbb{R}^{2n}} K(Qz, Qz') \Psi(z') dz' \quad (34)$$

By defining q and p in this way we have implicitly chosen a base for the phase space $z = (q_1, \dots, q_n, p_1, \dots, p_n)$, in which J takes the form

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (35)$$

With this we see from Eq. (29) that Poisson Kick Hamiltonians are indeed self-preserving

$$QJQ^T = (I_n \ 0_n) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} I_n \\ 0_n \end{pmatrix} = 0. \quad (36)$$

Noting that $\ker(QJ) = \mathbb{R}^n \times \{0\}^n$ it becomes apparent from Eq. (31), that any single-particle Hamiltonian H_0 that does not depend on p will keep the total Hamiltonian $H[\Psi] = H_K[\Psi] + H_0$ self-preserving.

The fact that Poisson-kicks are self-preserving has already been implicitly used by Cheng to develop an efficient operator splitting scheme for the Vlasov-Poisson equation [7].

SUMMARY & OUTLOOK

In this contribution we proposed an expansion procedure for the Vlasov equation and showed that it is possible to extract a class of linearly collective, self-preserving Hamiltonians from the first iteration this expansion. As already hinted, one possible way forward is to truncate the expansion at some order and suppress any remaining Ψ dependencies. The resulting Liouville equation can then be solved exactly via the method of characteristics and one might be able to estimate the error bounds resulting from that truncation.

APPENDIX

Let $u \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ be compactly supported and $v \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, then by integration by parts

$$\int_{\mathbb{R}^{2n}} \{u, v\} dz = \sum_{i,j=1}^{2n} J_{ij} \int_{\mathbb{R}^{2n}} \frac{\partial u}{\partial z_i} \frac{\partial v}{\partial z_j} dz \quad (37)$$

$$= - \sum_{i,j=1}^{2n} J_{ij} \int_{\mathbb{R}^{2n}} u \frac{\partial^2 v}{\partial z_i \partial z_j} dz = 0, \quad (38)$$

where the last equality is by the anti-symmetry of J_{ij} .

Let additionally $w \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. As the Poisson bracket obeys the Leibnitz rule, we see that

$$\int_{\mathbb{R}^{2n}} \{u, v\} w dz = \int_{\mathbb{R}^{2n}} \{u, vw\} - \{u, w\} v dz \quad (39)$$

$$= \int_{\mathbb{R}^{2n}} \{w, u\} v dz, \quad (40)$$

where in the last equality we have used the previous result, Eq. (38), and the antisymmetry of the Poisson bracket.

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