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FREQUENCY DEPENDENCE OF PLASMA CASCADE AMPLIFICATION*

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Abstract

By incorporating the longitudinal electric field reduction due to finite transverse beam size into the Poisson equation and solving the Hill's equation with time-dependent plasma frequency, we investigate how the amplification of a Plasma-Cascade Amplifier (PCA) depends on the spatial frequency of the density modulation.

INTRODUCTION

E A new type of amplifier, plasma cascade amplifier (PCA) has been proposed for a coherent electron cooling (CeC) system [1-3]. Previously, the 1D model for PCA assumes that the transverse distribution of the density perturbation in the electrons is uniform and consequently, the plasma frequency does not depend on the wavelength of the perturbation [1]. This assumption is valid if the longitudinal wavelength of the initial perturbation in the beam frame is much shorter than the transverse width of perturbation.

In this work, we explore the PCI gain at long wavelength by assuming the perturbation in electrons' density has non-uniform transverse profile. Specifically, we solve the 3D Poisson equation for given charge distribution (longitudinal sinusoidal, transversely Gaussian or Beer-can), average the electric field over the transverse plane and then apply it to 1-D Vlasov equation. Similar to the previous calculation in [1], the Vlasov equation can be reduced to a Hill's equation but the plasma frequency now depends on the longitudinal wavelength of the density perturbation in the electrons. By numerically solving the Hill's equation, we obtain the gain of a PCA as a function of spatial frequency, k_z.

EQUATION OF MOTION

For 1-D analysis, we treat each electron as a charge disc with certain charge distribution and consequently the evolution of the electrons' density perturbation in the beam frame are determined by the linearized 1-D Vlasov equation,

$$\frac{\partial}{\partial t} f_1(z, v_z, t) + v_z \frac{\partial}{\partial z} f_1(z, v_z, t) + \dot{v}_z \frac{\partial}{\partial v_z} f_0(z, v_z, t) = 0, \tag{1}$$

where v_z is the longitudinal velocity of the electrons, z is the longitudinal position along the bunch,

$$\dot{v}_z = -\frac{eE_z(z,t)}{m_e}, \tag{2}$$

is the longitudinal acceleration due to the longitudinal electric field $E_z(z,t)$, e is the absolute value of the charge of an electron and $f_1(z, v_z, t)$ is the density perturbation of the electrons in the longitudinal phase space and $f_0(v_z)$ is the unperturbed distribution of electrons. The longitudinal electric field due to density perturbation is determined by the Poisson equation

$$\nabla \cdot \vec{E} = -\frac{en(r,z,\theta,t)}{\varepsilon_0}, \tag{3}$$

where $n(r, z, \theta, t)$ is the local spatial density perturbation of the electrons. For the next step, we will try to solve Eq. (3) and since there is no operation in time t for Poisson equation, we will omit t from the variables in the bracket for now and will put it back when we solve the coupled-Poisson-Vlasov equation system. If we assume that the spatial distribution of the electron beam has cylindrical symmetry and define

$$n(r, z) = \rho_1(z) f_\perp(r), \tag{4}$$

Eq. (3) becomes

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\varphi(r,z)\right)\right] + \frac{\partial^2}{\partial z^2}\varphi(r,z) = -\frac{e}{\varepsilon_0}\rho_1(z) f_\perp(r), \tag{5}$$

where $\varphi(r, z)$ is the electric potential, $f_\perp(r)$ is the transverse surface density of electrons in unit of m^{-2} and $\rho_1(z)$ is perturbation of electron line density in unit of m^{-2}. Multiplying both sides of Eq. (5) by e^{-ikz} and integrating over z yields

$$\frac{\partial^2}{\partial r^2}\phi + \frac{1}{r}\frac{\partial}{\partial r}\phi - k^2\phi = f(r), \tag{6}$$

with
$$\phi(k, r) = \int_{-\infty}^{\infty} e^{-ikz}\varphi(r,z)dk, \tag{7}$$

$$f(r) = -\frac{e}{\varepsilon_0} f_\perp(r) \int_{-\infty}^{\infty} e^{-ikz}\rho_1(z)dz = -\frac{e}{\varepsilon_0} f_\perp(r)\tilde{\rho}_1(k), \tag{8}$$

and
$$\tilde{\rho}_1(k) = \int_{-\infty}^{\infty} e^{-ikz}\rho_1(z)dz. \tag{9}$$

The solution of Eq. (6) can be written as

$$\phi(r) = c_1 I_0(kr) + c_2 K_0(kr)$$

$$+\frac{1}{k}\int_{r_0}^{r} \frac{I_0(k\xi)K_0(kr) - K_0(k\xi)I_0(kr)}{I_0(k\xi)K_0{}'(k\xi) - K_0(k\xi)I_0{}'(k\xi)} \cdot f(\xi)d\xi, \tag{10}$$

where $I_0(x)$ and $K_0(x)$ are the modified Bessel functions. By applying the following boundary conditions, $\phi(\infty) = 0$ and $\lim_{r\to 0}\phi(r) \neq \infty$, the coefficient, c_1 and c_2 can be determined as

$$c_1 = 0, \tag{11}$$

and

$$c_2 = -\int_0^{\infty} \xi I_0(k\xi) \cdot f(\xi)d\xi. \tag{12}$$

Inserting Eq. (11) and Eq. (12) into Eq. 10 leads to

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$$\phi(r) = I_0(kr)\int_\infty^r \xi K_0(k\xi)\cdot f(\xi)d\xi - K_0(kr)\int_0^r \xi I_0(k\xi)\cdot f(\xi)d\xi. \quad (13)$$

The longitudinal electric field is obtained from

$$E_z = -\frac{\partial\varphi}{\partial z} = -\frac{i}{2\pi}\int_{-\infty}^{\infty} k\phi(k,r)e^{ikz}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_z(r,k)e^{ikz}dk,$$

with

$$\tilde{E}_z(r,k) = -ik\phi(r). \quad (14)$$

If we assume that the transverse distribution of the electron density perturbation is Gaussian, i.e.

$$f(r) = -\frac{e}{\varepsilon_0}\tilde{\rho}_1(k)\frac{1}{2\pi\sigma_r^2}exp\left(-\frac{r^2}{2\sigma_r^2}\right), \quad (15)$$

the longitudinal electric field averaged over the transverse plane reads

$$E_{z,avg}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_{z,avg}(k)e^{ikz}dk, \quad (16)$$

with

$$\tilde{E}_{z,avg} = 2\pi\int_0^\infty rf_\perp(r)\tilde{E}_z(r)dr = -\frac{ie\tilde{\rho}_1(k)}{4\pi\varepsilon_0 k\sigma_r^2}\cdot R(k\sigma_r), \quad (17)$$

and

$$R(k\sigma_r) \equiv 4k^2\sigma_r^2\int_0^\infty d\zeta\int_0^\zeta d\eta\cdot\zeta\eta I_0(k\sigma_r\eta)K_0(k\sigma_r\zeta)e^{\frac{\eta^2+\zeta^2}{2}}. \quad (18)$$

There are other forms of $R(k\sigma_r)$ such as

$$R(k\sigma_r) = k^2\sigma_r^2 e^{k^2\sigma_r^2}Ei(k^2\sigma_r^2), \quad (19)$$

and

$$R(k\sigma_r) = 2k\sigma_r\int_0^\infty \Phi(x)\sin(k\sigma_r x)\,dx, \quad (20)$$

with

$$\Phi(x) = \frac{1}{2}\left[\frac{x}{|x|} - \frac{x\sqrt{\pi}}{2}exp\left(\frac{x^2}{4}\right)erfc\left(\frac{|x|}{2}\right)\right], \quad (21)$$

and numerical evaluation verifies that they are equivalent to Eq. (18).

For uniform transverse charge distribution, instead of Eq. (15), we take

$$f(r) = -\frac{e}{\varepsilon_0}\tilde{\rho}_1(k)\frac{1}{\pi a^2}H(a-r), \quad (22)$$

where $H(x)$ is the Heaviside step function, a is the beam radius and the normalization of $f(r)$ is

$$2\pi\int_0^\infty rf(r)dr = -\frac{e}{\varepsilon_0}\tilde{\rho}_1(k). \quad (23)$$

Inserting Eq. (22) into Eq. (14) yields

$$\tilde{E}_z(r) = ik\frac{e}{\pi\varepsilon_0}\tilde{\rho}_1(k)\times\left[I_0(kr)\int_{r/a}^1 \eta K_0(ka\cdot\eta)\,d\eta + K_0(kr)\int_0^{r/a}\eta I_0(ka\cdot\eta)\,d\eta\right]. \quad (24)$$

Averaging Eq. (24) over the transverse plane leads to

$$\tilde{E}_{z,avg} = \frac{2\pi}{\pi a^2}\int_0^a rdr\tilde{E}_z(r) = i\frac{e}{\pi\varepsilon_0 ka^2}\tilde{\rho}_1(k)R(ka), \quad (25)$$

with

$$R(ka) \equiv \frac{4}{(ka)^2}\int_0^{ka} I_1(\tau)K_0(\tau)\tau^2 d\tau. \quad (26)$$

Figure 1 shows that for the same RMS beams size of 0.1 mm, the field reduction for the uniformly distributed beam is more significant than that for the electrons with Gaussian transverse spatial distribution.

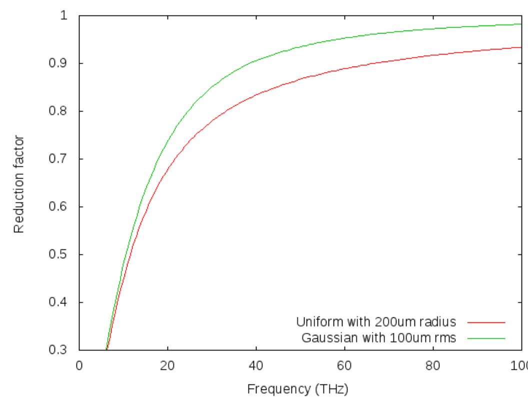

Figure 1: Reduction factor as calculated from Eq. (18) for the Gaussian transverse distributed electrons with RMS beam width of 0.1 mm and that calculated by Eq. (26) with beam radius of 0.2 mm.

Multiplying both sides of Eq. (1) by $exp(-ikz)$ and integrating over z yields

$$\frac{\partial}{\partial t}\tilde{f}_1(k,v_z,t) + ikv_z\tilde{f}_1(k,v_z,t) - \frac{e}{m_e}\tilde{E}_z(k,t)\frac{\partial}{\partial v_z}f_0(v_z) = 0, \quad (27)$$

with $\tilde{f}_1(k,v_z,t) \equiv \int_{-\infty}^{\infty}e^{-ikz}f_1(z,v_z,t)dz$. Multiplying both sides of Eq. (28) by $exp(ikv_z t)$ leads to

$$\frac{\partial}{\partial t}\left[e^{ikv_z t}\tilde{f}_1(k,v_z,t)\right] = \frac{e}{m_e}\tilde{E}_z(k,t)e^{ikv_z t}\frac{\partial}{\partial v_z}f_0(v_z). \quad (28)$$

Integrating Eq. (29) over time, t, leads to

$$\tilde{f}_1(k,v_z,t) = e^{-ikv_z t}\tilde{f}_1(k,v_z,0) + \frac{e}{m_e}\frac{\partial}{\partial v_z}f_0(v_z)\int_0^t\tilde{E}_z(k,t_1)e^{-ikv_z(t-t_1)}dt_1. \quad (29)$$

Replacing $\tilde{E}_z(k,t_1)$ with the averaged field for uniform distribution, Eq. (25), yields

$$\tilde{f}_1(k,v_z,t) = e^{-ikv_z t}\tilde{f}_1(k,v_z,0) + \frac{ie^2}{k\varepsilon_0 m_e}\frac{\partial}{\partial v_z}f_0(v_z)\int_0^t\frac{\tilde{\rho}_1(k,t_1)}{S^*(k,t_1)}e^{-ikv_z(t-t_1)}dt_1, \quad (30)$$

with

$$S^*(k,t) \equiv \frac{\pi a^2}{R(ka)}. \quad (31)$$

Integrating Eq. (30) over v_z gives

$$\tilde{\rho}_1(k,t) = \int_{-\infty}^{\infty}e^{-ikv_z t}\tilde{f}_1(k,v_z,0)dv_z - \frac{e^2}{\varepsilon_0 m_e}\int_0^t dt_1(t-t_1)\frac{\tilde{\rho}_1(k,t_1)}{S^*(t_1)}\left\{\int_{-\infty}^{\infty}f_0(v_z)e^{-ikv_z(t-t_1)}dv_z\right\}, \quad (32)$$

where we used the following relation

$$\tilde{\rho}_1(k,t) = \int_{-\infty}^{\infty}\tilde{f}_1(k,v_z,t)dv_z. \quad (33)$$

Assuming the unperturbed electron line density is ρ_0 and taking the following unperturbed velocity distribution,

$$f_0(v_z) = \frac{\rho_0\sigma_v}{\pi}\frac{1}{\sigma_v^2+v_z^2}, \quad (34)$$

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Eq. (32) becomes

$$\tilde{\rho}_1(k,t) = \int_{-\infty}^{\infty} e^{-ikv_z t}\tilde{f}_1(k,v_z,0)dv_z - \frac{e^2\rho_0}{\varepsilon_0 m_e}\int_0^t dt_1(t-t_1)\frac{\tilde{\rho}_1(k,t_1)}{S^*(t_1)}exp[-|k|\sigma_v(t-t_1)]. \quad (35)$$

Taking second time derivative of Eq. (36) yields

$$\frac{d^2}{dt^2}\tilde{R}_1(k,t) + \frac{e^2\rho_0}{\varepsilon_0 m_e S^*(t)}\tilde{R}_1(k,t) = -\frac{e^2\rho_0}{\varepsilon_0 m_e S^*(t)}\int_{-\infty}^{\infty} e^{-ikv_z t}\, e^{\widetilde{|k|\sigma_v t}} f_1(k,v_z,0)dv_z, \quad (36)$$

where

$$\tilde{R}_1(k,t) \equiv \tilde{\rho}_1(k,t)\, e^{|k|\sigma_v t} - \int_{-\infty}^{\infty} e^{-ikv_z t}\, e^{\widetilde{|k|\sigma_v t}} f_1(k,v_z,0)dv_z. \quad (37)$$

If we further assuming the initial perturbation has the following form

$$f_1(z,v_z,0) = \frac{\rho_1(z,0)\sigma_z}{\pi}\frac{1}{\sigma_z^2+v_z^2}, \quad (38)$$

Eq. (37)becomes

$$\frac{d^2}{dt^2}\tilde{Q}_1(k,t) + \frac{e^2\rho_0 R(k,t)}{\varepsilon_0 m_e S(t)}\tilde{Q}_1(k,t) = 0, \quad (39)$$

where $S(t) = \pi a^2$, $R(k,t)$ is to be calculated from Eq. (26) and

$$\tilde{Q}_1(k,t) \equiv \tilde{\rho}_1(k,t)\, e^{|k|\sigma_z t}\ . \quad (40)$$

For electrons with Gaussian transverse distribution, it is easy to obtain an equation with identical form of Eq.(39), except $S(t) = 4\pi\sigma_r^2$ and $R(k,t)$ should be calculated from Eq.(18).

PCA GAIN

Evolution of the beam cross section, $S(t)$, is determined by the envelope equation. If we use the normalized variable as defined in [2], the coupled equation of motion of the PCA system in the lab frame can be written as

$$\hat{a}'' = k_{sc}^2\hat{a}^{-1} + k_\beta^2\hat{a}^{-3}, \quad (41)$$

and

$$\tilde{Q}_1''(k,s) + \frac{2k_{sc}^2 R(k,s)}{\hat{a}^2}\tilde{Q}_1(k,s) = 0 \quad (42)$$

with $\hat{a}(t) = a(t)/a_0$, $k_{sc} = l\sqrt{2I_0/(\beta^3\gamma^3 I_A a_0^2)}$ and $k_\beta = \varepsilon l/a_0^2$. Figure 2 shows the PCA gain as a function of frequency, $f_{lab} = \gamma kc/(2\pi)$, with RMS beam width at waist of 0.1 mm for the Gaussian distribution and beam radius of 0.2 mm for the uniform distribution.

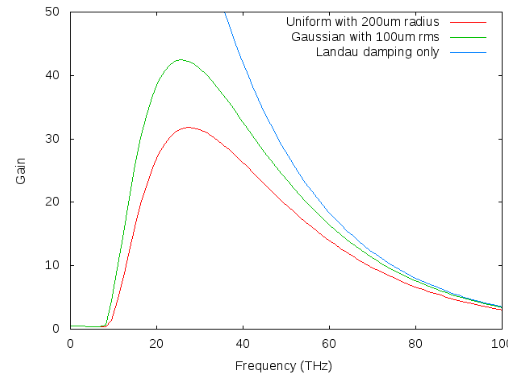

Figure 2: Comparison of PCI amplification gain for two types of transverse charge distribution of the modulation: uniform (red) and Gaussian (green). We take $\sigma_v/c = 2\times10^{-4}$, $k_{sc} = 3.6$ and $k_\beta = 7$ for the plots.

SUMMARY

By incorporating the reduction factor for the longitudinal electric field into the Hill's equation, we obtained the frequency dependence of the PCA gain, which shows similar behaviour at low frequency as what obtained from 3D simulation (see Fig. 7 of [3]).

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