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SYMPLECTIC TRACKING THROUGH FIELD MAPS

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Abstract

For studies of beam dynamics with complicated geometries of the fields, it is necessary to track particles using field maps, instead of an analytic representation of the fields which is typically not available. These field maps come about while designing elements such as realistic magnets or radiofrequency cavities, and represent the field geometry on a mesh in space. However, simple interpolation of the fields from the field maps does not guarantee that the resulting tracking scheme satisfies the symplectic condition. Here we present a general method to decompose the field-map potential in sum of interpolating functions that produces, by construction, a symplectic integrator.

INTRODUCTION

Tracking codes frequently rely on simplified, analytic representations of common elements, such as rf cavities or quadrupoles, with multi-mode representations to generalize and include, for example, sextupolar terms in an imperfect quadrupole. This representation guarantees an analytic, functional form of the potentials required for symplectic tracking, allowing the code to analytically represent those gradients and assuring the tracking method satisfies the symplectic condition. For more realistic designs of, for example, rf cavities, the design fields may be represented as field maps – representing the fields on a mesh. In this case, tracking requires interpolating between the mesh points. However, there is no guarantee that this interpolation can capture the gradients exactly, which is required for preserving the symplectic condition.

We present a generic approach to represent the field maps using local basis functions to define the interpolation. We apply this approach to a generic integrator through *s*- and *t*-dependent fields described by Wu *et al.* [1]. Finally, we demonstrate this method on the field map of an quadrupole-with errors in a FODO cell repeated multiple times to verify the symplectic condition.

A FIELD MAP HAMILTONIAN

To derive a symplectic tracking algorithm for particle trajectories in potentials specified by field maps, we start at the level of the Lagrangian. If we define four-velocities and potentials as

$$\begin{aligned} r^\alpha &= (ct, \vec{r}) = (ct, x, y, s) \\ v^\mu &= (c\gamma, \vec{v}) = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{ds}{d\tau}\right) \\ v_\mu &= \left(-c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{ds}{d\tau}\right) \\ A^\mu &= \left(\frac{\phi}{c}, A_x, A_y, A_s\right), \end{aligned} \tag{1}$$

where t is the lab-frame time and τ proper time, and s is the beam axis. Then the Lagrangian of interest to us for s-based tracking is [2]

$$L = -mc^2\sqrt{v_\mu v^\mu} + qv^\mu A_\mu, \tag{2}$$

where q is species charge. We can consider this Lagrangian in the context of an ensemble

$$L = \int d\vec{r}\, f(\vec{r}) \left\{-mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}} - q\phi(\vec{r}) + \frac{q}{c}\vec{v}\cdot\vec{A}\right\}, \tag{3}$$

where f is the phase space density. For an ensemble of single particles, as we would trace in a single-particle tracking code, the phase space density is a Dirac delta-function:

$$f(\vec{r}, \vec{r}') = \sum_j \delta(\vec{r} - \vec{r}_j)\delta(\vec{r}' - \vec{r}'_j). \tag{4}$$

Local Basis Representation for Field Maps

Suppose we know the fields for each component j of the potentials, A^j_σ, from the field map for a set of points $\{R_\sigma\}$. To build our Hamiltonian, we require functions which interpolate between those points. We will define an arbitrary *valid interpolation* as one for which

$$A^\mu(\vec{r}) = \sum_a a^\mu_a \Psi(\vec{r} - \vec{r}_a), \tag{5}$$

such that

$$\sum_a a^\mu_a \Psi(\vec{R}_a - \vec{r}_a) = A^\mu_\sigma \tag{6}$$

for each σ. We will leave the choice of Ψ generic for now, although some representations will be more convenient than others based on the structure of the field maps.

Inserting this field definition into the Lagrangian gives a field map Lagrangian for particle j as

$$L_j = -mc^2\sqrt{1 - \frac{\vec{v}_j^2}{c^2}} - q\varphi(\vec{r}_j) + \frac{q}{c}\vec{v}\cdot\vec{\mathcal{A}}(\vec{r}_j), \tag{7}$$

where

$$\mathcal{A}^\mu(\vec{r}_j) = \sum_a a^\mu_a \Psi(\vec{R}_a - \vec{r}_j). \tag{8}$$

At this point, we can define analytically any required derivative of $\vec{A}$ or ϕ in terms of derivatives of this valid interpolation.

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Field Map Hamiltonian

With a valid field map, we can compute a Hamiltonian for each particle we are tracking, through the usual method of defining a canonical momentum

$$\vec{p}_j = m\vec{v}_j\sqrt{1-\frac{\vec{v}_j^2}{c^2}} + \frac{q}{c}\vec{\mathcal{A}}(\vec{r}_j), \tag{9}$$

and computing the Hamiltonian from the Legendre transform [3]

$$\mathcal{H}_j = c\sqrt{\left(\vec{p}_j - \frac{q}{c}\vec{\mathcal{A}}(\vec{r}_j)\right)^2 + m^2c^2} + q\varphi(\vec{r}_j). \tag{10}$$

This is the Hamiltonian for tracking particles in time, and we have so far considered only fields that are static in time[1].

In most accelerator applications, we are interested in tracking in either s-based on Frenet–Serret coordinates [4], and we may apply that same computation to this Hamiltonian. For this example we will consider a purely linear accelerator, with no dipoles, to illustrate the concept. As most accelerator tracking codes use the longitudinal coordinate, s, as the independent variable, the generator of s translations is $-p_z \equiv \mathcal{P}$. Defining $-p_\tau = \mathcal{H}/c$ and $\tau = ct$ as canonically conjugate coordinates, we can compute the Hamiltonian for s-translations:

$$\mathcal{P} = \sqrt{\left(p_\tau + \frac{q}{c}\varphi(\vec{r}_\perp, s)\right)^2 - \left(\vec{p}_\perp - \frac{q}{c}\vec{\mathcal{A}}_\perp(\vec{r}_\perp, s)\right)^2 - m^2c^2} + \frac{q}{c}\mathcal{A}_s(\vec{r}_\perp, s). \tag{11}$$

This Hamiltonian is of a form conducive to the symplectic integrator derived by Wu *et al.* [1].

Most symplectic integrators use a split map approach [5], based on the Lie algebraic formalism, e.g:

$$\mathcal{M}_{(s\to s+\Delta s)} \approx e^{-:H_0:\Delta s/2} e^{-:H_1:\Delta s} e^{-:H_0:\Delta s/2}. \tag{12}$$

This requires exact evaluation of the Poisson brackets, which have partial derivatives of the coordinates. This means that a "kick" needs to be computed as the exact gradient of a scalar function.

$$p^f = p^i + \nabla_q V\left(q^i\right)\Delta s. \tag{13}$$

For 2- and 3-D field maps, interpolating the fields does not guarantee the resulting kick is the exact gradient of a scalar function.

$$\vec{F} = \sum_{i,j,k} w_{i,j,k}\vec{f}_{i,j,k} \stackrel{?}{=} \nabla_q V(q). \tag{14}$$

In other words, symplectic integration using field maps requires an exactly differentiable interpolation between grid points in the field map.

$$\frac{\partial A}{\partial x}(\vec{r}) = \sum_\sigma a_\sigma \frac{\partial \Psi_\sigma}{\partial x}(\vec{r}). \tag{15}$$

[1] The generalization to time-varying fields, such as RF cavity modes, is a subject of future work

Figures 1 and 2 illustrate field maps and gradient maps generated for this study. These maps are generated from analytical expressions for quadrupole potentials

$$A_s = k(x^2 - y^2). \tag{16}$$

New random noise is added for each longitudinal slice, giving a rudimentary approximation of a measured 3D fieldmap (without fringe fields). Each longitudinal slice uses a 2D cubic spline interpolation, with linear interpolation between longitudinal slices.

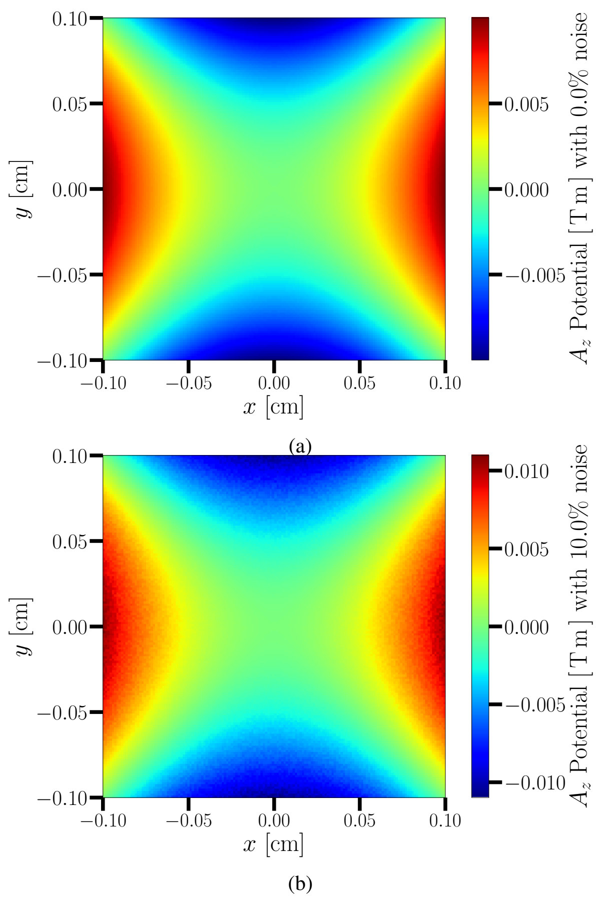

Figure 1: Quadrupole field maps with no noise (a) and 10% random noise (b). These are generated from analytical expressions for quadrupole potentials, given random noise to approximate a measured field-map, then interpolated quadratically.

EXAMPLE: FODO CELL WITH QUADRUPOLE FIELD MAPS AND NOISE

The Hamiltonian we use is of the form

$$\mathcal{H} = \frac{1}{2}\vec{p}_\perp^2 + K\mathcal{A}_s(x, y, s) \tag{17}$$

for simplicity, as the symplectic nature of the integrator is determined purely from taking the gradients exactly in the integration kicks.

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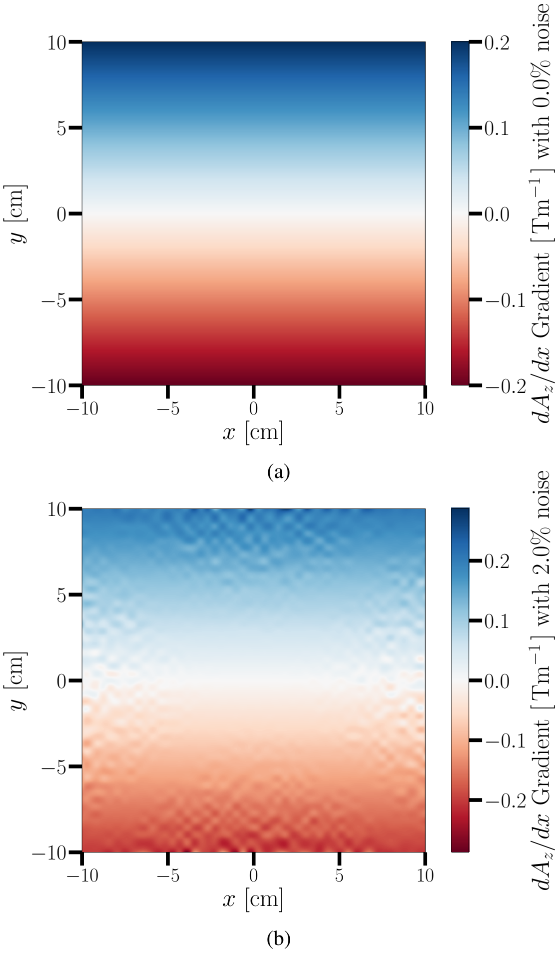

Figure 2: Quadrupole gradient maps with no noise (a) and 2% random noise (b), generated and interpolated in the same manner as the field in Fig. 1.

To demonstrate this technique, we use a second-order drift-kick integrator using an exact derivative prescribed in Eq. (13). This derivative is computed using cubic splines for each longitudinal slice, and interpolating between slices linearly. This linear interpolation does not affect the symplecticity, as no derivatives with respect to s are required.

First, we verify that the method is symplectic by checking its stability through one million turns in a simple FODO cell (Fig. 3a). Here we see a clean elliptical trajectory, as we would expect for a particle in a purely linear FODO cell.

We then test the same integrator with an added 5% noise between longitudinal field map slices (Fig. 3b). In the latter example, unphysical high-order nonlinearities are introduced, but the Hamiltonian dynamics are preserved without the unphysical heating or cooling of the trajectory that is usually seen in non-symplectic integrators. This is suggestive that this approach is well-suited for symplectic tracking broadly.

DISCUSSION

We have presented a generic approach to building a symplectic integrator from field map data of the vector potentials.

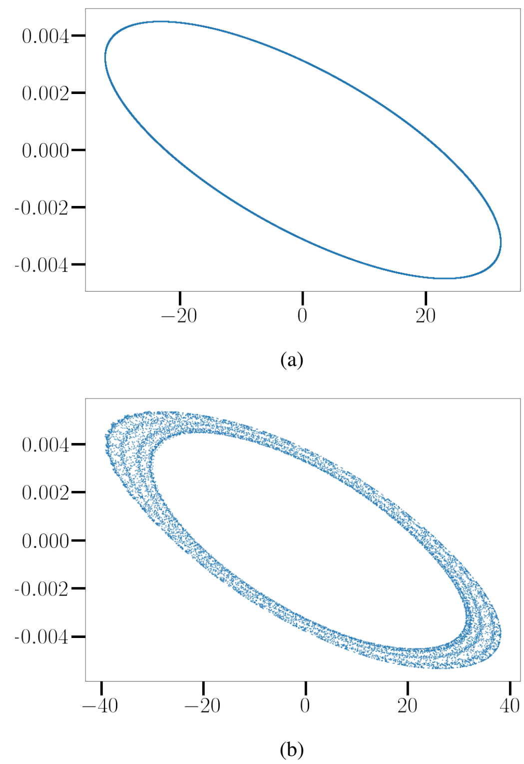

Figure 3: Single particle tracking through an interpolated-field map FODO cell: (a) one million turns with zero noise maps; (b) 10,000 turns with 5% noise.

This approach revolves around ensuring that all derivatives or anti-derivatives that appear in the integration scheme are exact, and is best achieved using differentiable interpolation functions such as spline functions rather than the usual finite differencing schemes. We have demonstrated this technique for the simple example of a paraxial FODO cell, finding that even with considerable noise levels in the computed magnetic potential we still preserve Hamiltonian structures. This work is suggestive of a new path for how tracking codes handle field maps.

REFERENCES

[1] Y. K. Wu, E. Forest, and D. S. Robin, "Explicit symplectic integrator for s-dependent static magnetic field," *Phys. Rev. E*, vol. 68, p. 046 502, 2003, `doi:10.1103/physreve.68.046502`

[2] L. D. Landau, *The classical theory of fields*. Amsterdam, Netherlands: Elsevier, 2013.

[3] A. Wolski, *Beam dynamics: In High Energy Particle Accelerators*. London, UK: Imperial College Press, 2014.

[4] S. Y. Lee, *Accelerator Physics*. Hackensack, NJ, USA: World Scientific, 2011.

[5] A.J. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics, `https://www.physics.umd.edu/dsat/dsatliemethods.html`.