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LINEAR COUPLING AND ADIABATICITY OF EMITTANCE EXCHANGE

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Abstract

In circular accelerators, crossing the coupling resonance induces the exchange of the transverse emittances, provided the process is adiabatic. In this paper, we introduce a theoretical framework to analyse the resonance-crossing process, based on Hamiltonian mechanics, which is capable of explaining all the features of the emittance exchange process.

INTRODUCTION

The impact of linear coupling on transverse betatron motion has been extensively studied as it has an impact already on the linear dynamics. In 2001, the phenomenon of emittance exchange due to dynamic crossing of the coupling resonance was studied [1], with further results reported in 2007 [2], in which it is mentioned that a full emittance exchange happens if the resonance crossing is adiabatic, and an adiabatic condition is given. This research has opened a new domain of investigations and a recent paper addressed the same topic with the goal to develop a complete theory of the emittance exchange process [3].

In recent years, there has been intense theoretical efforts to study in detail the phenomenon of resonance crossing in one degree-of-freedom (1DoF) Hamiltonian systems in view of devising novel beam manipulations [4–9]. This culminated in the proposal and final implementation of the CERN PS Multi-Turn Extraction (MTE) as an operational means to provide an optimized extraction technique based on nonlinear beam dynamics [10–16]. It is worth stressing that the mathematical framework for these studies is the theory of adiabatic invariance for Hamiltonian systems.

This framework provides also the natural way of addressing the analysis of the resonance crossing in the presence of linear coupling. In this paper, we show how all observations reported in previous works, in particular in [2, 3], find a clear explanation using the results of adiabatic theory. The analysis is also extended to the case in which nonlinear detuning with amplitude is present.

THE HAMILTONIAN MODEL AND ITS DYNAMICS

Following the treatment used in Refs. [17–19], we consider the Hamiltonian

$$H(p_x,p_y,x,y) = \frac{p_x^2+p_y^2}{2} + \frac{1}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + 2qxy\right), \tag{1}$$

where $q = -\sqrt{\beta_x\beta_y}\,\hat{q}$, and the coefficient $\hat{q}$ is defined as

$$\hat{q} = \frac{1}{2B\rho}\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x}\right), \tag{2}$$

and represents the strength of a skew quadrupole on the betatron dynamics. In the following, the notation z will be used to denote either the coordinate x or y.

We consider the adiabatic crossing of the linear coupling resonance, namely $\omega_x - \omega_y = 0$, when the frequencies are slowly modulated, and we define

$$\delta(\lambda) = \omega_x(\lambda) - \omega_y(\lambda) \qquad \delta_2(\lambda) = \omega_x^2(\lambda) - \omega_y^2(\lambda), \tag{3}$$

with $\lambda = \epsilon t$, $\epsilon \ll 1$, and ϵ is the adiabatic parameter that describes the resonance crossing process. Without loss of generality, $\delta(\lambda)$ is defined by a linear function that varies from positive to negative values (or vice versa) crossing zero.

The eigenvalues of the potential matrix are given by

$$\omega_{1,2}^2 = \frac{\omega_x^2+\omega_y^2 \pm \sqrt{\delta_2^2(\lambda)+4q^2}}{2}, \tag{4}$$

and the corresponding eigenvectors are

$$v_1(\lambda) = c_1\left(\frac{\delta_2(\lambda)+\sqrt{\delta_2^2(\lambda)+4q^2}}{2}, q\right), \tag{5}$$

where c_1 is a normalising constant and $v_2(\lambda) \perp v_1(\lambda)$.

For $q \ll 1$ and $\delta_2(\lambda) > 0$, one has $v_1 \to e_x$ and $v_2 \to e_y$, where e_x, e_y are the unit vectors defining the horizontal and vertical planes. When $\delta_2(\lambda) = 0$, i.e. $\omega_x(\lambda) = \omega_y(\lambda)$, then v_1 and v_2 define the two bisectors of the two angles defined by the horizontal axis and the positive vertical axis, whereas when $|q| \ll 1$ and $\delta_2(\lambda) < 0$, then $v_1 \to e_y$ and $v_2 \to -e_x$. Therefore, the passage through the resonance $\omega_x - \omega_y$ implies an exchange of the direction of the eigenvectors.

Note that the difference resonance cannot be crossed by $\omega_{1,2}$ as the eigenvalues cannot get closer than $(\omega_1^2-\omega_2^2)_{\min} = 2|q|$ as it is well-known (see, e.g. Refs. [2, 3] and references therein). Hence, in the physical co-ordinates, the coupling resonance can be crossed, but the tunes are not the eigenvalues of the system. On the other hand, in the coordinate system of the eigenvalues the resonance cannot be crossed, although the eigenvalues are the proper quantities to describe the dynamics.

After a sequence of transformations, the introduction of a slow phase $\phi_a = \theta_x - \theta_y$, and the application of a perturbative approach, averaging over the fast-evolving angle

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$\phi_b = \theta_y$, it is possible to study the resonance-crossing process for the original Hamiltonian (1). We consider H of the following form

$$H(\phi, J, \lambda) = \delta(\lambda) J + q\sqrt{(1-J)J}\sin\phi\,, \tag{6}$$

where, without loss of generality, we have re-scaled the action according to $J = J_a/J_b$ so that $J = 0$ and $J = 1$ are singular lines for the Hamiltonian (6). We also defined $\phi = \phi_a + \pi/2$, and then replaced $\delta(\lambda) \rightarrow \delta(\lambda)\sqrt{\omega_x(\lambda)\omega_y(\lambda)}/J_b$, which corresponds to a global rescaling of the Hamiltonian.

Note that the Hamiltonian (6) has the form

$$H(\phi, J, \lambda) = \epsilon t J + q H_1(J, \phi)\,, \tag{7}$$

for which the equations of motion are

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -q\frac{\partial H_1}{\partial \phi} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \epsilon t + q\frac{\partial H_1}{\partial J}\,. \tag{8}$$

By introducing a new time $\bar{t} = q\,t$, Eq. (8) can be rewritten in the following form

$$\frac{\mathrm{d}J}{\mathrm{d}\bar{t}} = -\frac{\partial H_1}{\partial \phi} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\bar{t}} = -\frac{\epsilon}{q^2}\bar{t} + \frac{\partial H_1}{\partial J}\,. \tag{9}$$

Thus, the small parameter characterising the adiabaticity is $\bar{\epsilon} = \epsilon/q^2$, and the new slow time is $\bar{\lambda} = (\epsilon/q^2)\bar{t}$. Note also that this approach can be extended to the case in which $\delta(\lambda)$ is a nonlinear function of λ. If we choose a polynomial expression, e.g. $\delta(\lambda) \propto (\lambda - \lambda_c)^{2n+1}$, where λ_c represents the time of the resonance crossing, it is easy to show that the small parameter characterising the adiabaticity is $\bar{\epsilon} = \epsilon/q^{\frac{2n+2}{2n+1}}$, and the exponent tends to 1 when $n \to \infty$. The level curve that reaches $J = 1$ at $\phi = 0$ and $\phi = \pi$ is a critical one and is tangent to the $J = 1$ curve, but it is not a singularity of the dynamics.

In Fig. 1, the phase-space portraits of the Hamiltonian of Eq. (6) (assumed to be frozen, i.e. with λ constant) are shown, for $q = 1$ and three values of δ, namely $1, 0, -1$ for the left, centre, and right plot, respectively.

The red curves represent the critical curve, which is also called *coupling arc* in Refs. [18, 19]. In the left plot ($\delta = 1$), two separated islands are visible, whose areas increase as δ decreases to zero. Furthermore, there exists a region of *separatrix curves* around the islands, tangent to the singular lines $J = 0$ and $J = 1$. When $\delta = 0$ (centre plot), the islands have maximal area, with a pseudo-separatrix that connects the singular line through the vertical line $\phi = \pi$. Finally, a symmetric situation when $\delta < 0$ is visible in the right plot.

It should be noted that this phase space is topologically a sphere, the two singular lines being identified with the north and the south pole.

The prototype Hamiltonian to study the emittance exchange process can be written in the normal-mode space

$$H(\phi, J, \lambda) = \gamma(\lambda) J + \epsilon\sqrt{(1-J)J}\sin\phi\,, \tag{10}$$

where we assume $J_2 = 1$, so that $J = 0$ and $J = 1$ are singular lines for the Hamiltonian, and $\gamma(\lambda) = O(q^2)$ for $\delta(\lambda) \to 0$. The action-angle variables are analytic for $\gamma(\lambda) \to 0$ and the Hamiltonian is analytic on the sphere. It is then possible to apply the Theorem reported in Ref. [20] to the Hamiltonian $H(\phi, J, \lambda)$ to state that the change of the action for a given orbit of the system is exponentially small, i.e. $\Delta J = O(\exp(-c/\epsilon))$ with c a positive constant, when λ varies, which corresponds to the crossing of the original difference resonance.

The same remarks made for the Hamiltonian of Eq. (6) about the re-scaled adiabaticity parameter hold also for (10). Hence, $\Delta J = O\big(\exp(-c\,q^2/\epsilon)\big)$ in case of a resonance crossing linear in λ, or $\Delta J = O\big(\exp\big(-cq^{\frac{2n+2}{2n+1}}/\epsilon\big)\big)$ in case of a nonlinear crossing of the resonance. Note that a nonlinear resonance crossing is more advantageous in terms of adiabaticity of the process with respect to a linear one.

Detuning with Amplitude

In presence of detuning with amplitude generated by nonlinearities, the dynamics is governed by the Hamiltonian of Eq. (1) plus the term [17]

$$\begin{aligned} H_{\mathrm{det}}(p_x, p_y, x, y) = {} & \alpha_{xx}\left(\frac{x^2+p_x^2}{2}\right)^2 + \\ & + 2\alpha_{xy}\left(\frac{x^2+p_x^2}{2}\right)\left(\frac{y^2+p_y^2}{2}\right) + \alpha_{yy}\left(\frac{y^2+p_y^2}{2}\right)^2 . \end{aligned} \tag{11}$$

The r.h.s. of Eq. (11) may generate hyperbolic fixed points in phase space [21], which imply the existence of a separatrix that introduces a singularity in the phase-space structure and hence alters the character of the dynamics. In particular, the nice property about the exponentially-small change of J, linked to the analyticity of the dynamics of (10), is lost.

SIMULATION RESULTS

Numerical simulations have been performed using map models (representing a system made of a FODO cell and a skew quadrupole, possibly with an octupolar term [21]).

The dependence of the emittance-exchange phenomenon on the adiabaticity of the resonance-crossing process was evaluated. In the simulations, ω_y has been varied while keeping ω_x constant. Thus, $\delta(\lambda) = \omega_x - \omega_y(\lambda)$ is varied from a negative to a positive value passing through zero. As a figure of merit for the exchange, we used the function P_{na}, introduced in [3], which is defined as

$$P_{\mathrm{na}} = 1 - \frac{\langle I_{x,\mathrm{f}}\rangle - \langle I_{x,\mathrm{i}}\rangle}{\langle I_{y,\mathrm{i}}\rangle - \langle I_{x,\mathrm{i}}\rangle}\,, \tag{12}$$

where $I_{z,\mathrm{i}}$ and $I_{z,\mathrm{f}}$ are the initial and final linear action variables, respectively. P_{na} is zero when a perfect exchange is attained and one when no exchange occurs.

The evolution of a set of initial conditions, representing a beam exponentially distributed in I_x, i.e. $\rho(I_x) = (N_0/\langle I_x\rangle)\exp(-I_x/\langle I_x\rangle)$, has been computed,

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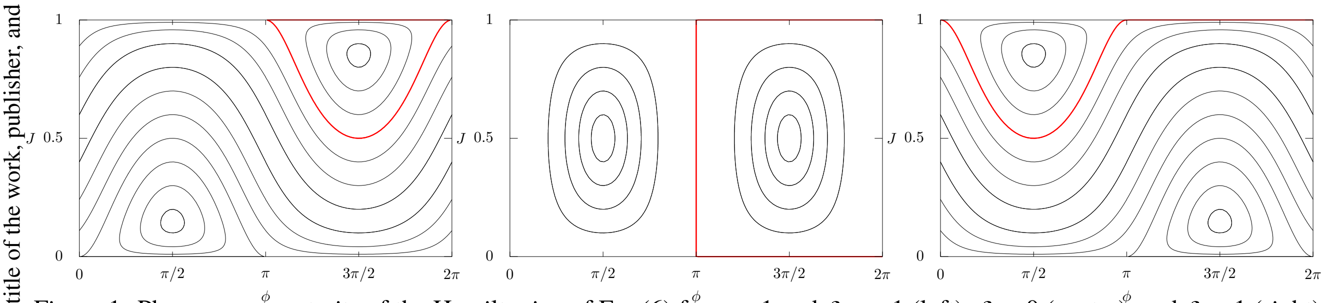

Figure 1: Phase-space portraits of the Hamiltonian of Eq. (6) for $q = 1$ and $\delta = -1$ (left), $\delta = 0$ (centre), and $\delta = 1$ (right) in action-angle co-ordinates (ϕ, J). The red line represents the critical curve (the so-called *coupling arc*).

while varying ω_y in the fixed interval $\omega_{y,\mathrm{i}} = 2.5$ and $\omega_{y,\mathrm{f}} = 2.7$ over a given time interval N. We expect that $\langle I_x \rangle$ becomes $\langle I_y \rangle$ after the resonance crossing. What we observe in Fig. 2 is a clear exponential dependence of P_{na} as a function of $1/\epsilon$, in evident agreement with the findings of Ref. [3] and with the qualitative discussion carried out previously.

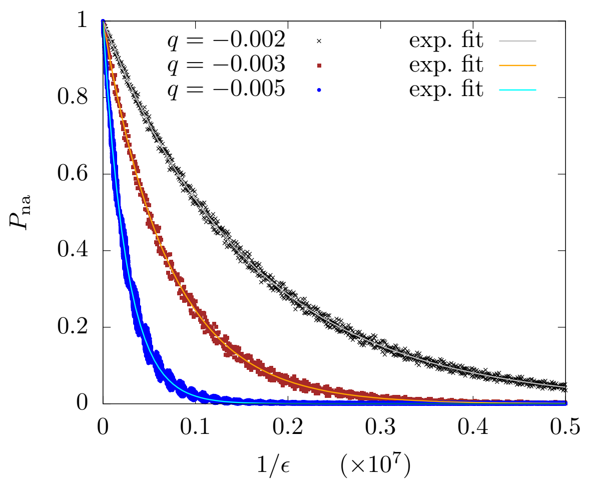

Figure 2: Evolution of P_{na} vs $1/\epsilon$ for an exponential distribution of initial conditions, and different values of q. Exponential fits are also presented. The linear map has been used, with parameters $\omega_x = 2.602$, $\omega_{y,\mathrm{i}} = 2.5$, $\omega_{y,\mathrm{f}} = 2.7$, and a set of initial conditions with $\langle I_{x,\mathrm{i}} \rangle = 10^{-4}$, $\langle I_{y,\mathrm{i}} \rangle = 4 \times 10^{-4}$.

The exponential behaviour of P_{na} features a clear dependence on q. However, an oscillatory behaviour is also observed due to the neglected terms $O(q^2)$ (see [21] for more detail). These scaling laws are not connected with the features of the distribution of initial conditions, but rather to the individual orbits of the Hamiltonian system.

We studied also the impact of detuning with amplitude on the adiabaticity of the emittance exchange using a map model with a normal octupole with normalised strength $k_3 = K_3 \beta_x^2/6$, setting $\chi = 1$, and simulating the resonance-crossing process in the same way as in absence of detuning.

For Gaussian distributions of initial conditions corresponding to different emittances in x and y, P_{na} has been evaluated for different values of k_3. The essential difference between the linear and nonlinear cases is clearly visible when investigating the dependence of P_{na} on the adiabatic parameter ϵ. This is shown in Fig. 3: the exponential function is lost and is replaced by a power-law for P_{na}.

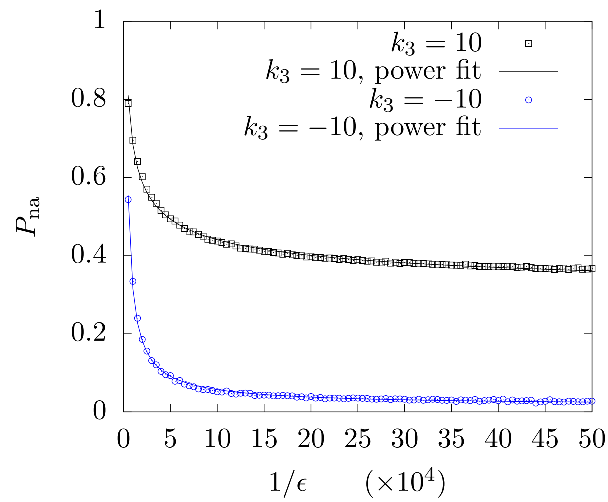

Figure 3: Dependence of P_{na} on ϵ for $k_3 = 10$ and $k_3 = -10$. A power-law dependence $P_{\mathrm{na}} = a\epsilon^b + c$, is fitted and the results also shown ($b = -0.381(9)$ for $k_3 = 10$ and $b = -0.844(9)$ for $k_3 = -10$).

In presence of nonlinear detuning with amplitude, it is possible to devise a scaling law linking q and ϵ. It is straightforward to conclude that P_{na} is linked to the change of the invariants during the crossing process. Therefore, the scaling law $P_{\mathrm{na}} = a\epsilon^{b(k_3,q)} + c(k_3, q)$, which has been analysed in Fig. 3, gives rise to the following relationship

$$\ln \epsilon = \frac{\text{const.} + c(k_3, q)}{a\, b(k_3, q)}, \tag{13}$$

which should be fulfilled to keep constant the change of the invariant. The essential difference with respect to what has been found in the absence of nonlinear detuning is apparent.

CONCLUSIONS

The Hamiltonian theory of the dynamic crossing of the coupling resonance has been presented and discussed, considering not only the linear, but also the nonlinear case. Key relationships between the system parameters and the adiabaticity condition of the resonance-crossing process have been derived. The observed phenomenology has been successfully explained.

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