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ALPHA BUCKETS IN LONGITUDINAL PHASE SPACE: A BIFURCATION ANALYSIS

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Abstract

At HZB's BESSY II and MLS facilities we have the ability to tune the momentum compaction factor α up to second non-linear order. The non-linear dependence $\alpha(\delta)$ brings qualitative changes to the longitudinal phase space and introduces new fix points $\alpha(\delta) = 0$ which produce the so-called α-buckets. We present with this paper an analysis of this phenomena from the standpoint of bifurcation theory. With this approach we were able to characterize the nature of the fix points and their position in direct dependence on the tunable parameters. Furthermore, we are able to place stringent conditions onto the tunable parameters to either create or destroy α-buckets.

INTRODUCTION

The operation of synchrotron storage rings in the so called quasi-isochronous mode is able to produce shorter bunch lengths leading to wavelengths in the THz radiation regime. This operation is possible due to the tweaking of the lattice optical parameters to lower the momentum compaction factor α. In the low-α mode the linear approximation of the momentum compaction factor is no longer valid and a higher order expansion is necessary. Placement of sextupole and octupole magnets then enables to tune the first two non-linear orders of α as has been successfully realized at the MLS [1] and DLS [2].

We neglect path lengthening due to betatron oscillation by assuming a suppressed transverse chromaticity to obtain a 1-dimensional model and explore the richness of the Hamiltonian to draw a bifurcation diagram. Furthermore, we find Hopf bifurcations occurring to produce further periodic orbits in the longitudinal phase space. We hope to present this theoretical classification of α-buckets to show the insight bifurcation theory can provide in manipulating non-linear beam dynamics.

LONGITUDINAL HAMILTONIAN

The length of the closed orbit in an synchrotron storage ring can be written as

$$L = L_0(1 + \alpha\delta), \tag{1}$$

where L_0 is the length of the reference orbit, δ the momentum deviation and $\alpha = \alpha(\delta)$ the momentum compaction factor which is momentum dependent. We have the typical expansion up to second order

$$\alpha(\delta) = \alpha_0 + \alpha_1\delta + \alpha_2\delta^2, \tag{2}$$

which we can control at the MLS. From [3, Eq. (5.6)] we use the derived Hamiltonian

$$H = -A\delta^2\left(\frac{\alpha_0}{2} + \frac{\alpha_1}{3}\delta + \frac{\alpha_2}{4}\delta^2\right) - B\cos\phi, \tag{3}$$

where A is a fixed property of the lattice, $B = B(U_0)$ can be tuned by the RF-Voltage U_0 and ϕ is the RF-cavity phase. The equations of motion then read

$$\begin{aligned}\dot{\phi} &= -A\delta(\alpha_0 + \alpha_1\delta + \alpha_2\delta^2)\\ \dot{\delta} &= -B\sin\phi.\end{aligned} \tag{4}$$

BIFURCATION THEORY

The general concern of bifurcation theory is how qualitative properties of the physical system change with respect to the underlying parameter space. For example by tweaking the above parameters in our Hamiltonian, Eq. (3), we can change the nature of fixpoints from stable to unstable and create or annihilate periodic solutions. Fixpoints are points (ϕ_*, δ_*) in phase space such that

$$\begin{aligned}\dot{\phi}(\phi_*, \delta_*) &= 0\\ \dot{\delta}(\phi_*, \delta_*) &= 0.\end{aligned} \tag{5}$$

We can then employ perturbation theory around the fixpoints to deduce the nature of the fixpoints, i.e. if they are stable (have periodic solutions around them) or unstable (have hyperbolic behaviour in their neighborhood). A detailed treaty on classification and the tools to analyze such behaviour can be found in [4].

APPLICATION TO ALPHA BUCKETS

In our case finding fixpoints is quite easy since there are no cross-terms in the Hamiltonian. We have two fixpoints $\phi_* = 0, \pi$ on which we will focus first. The expansions around the two fixpoints differ only by an overall sign change, which we will absorb into the parameter B and obtain

$$\begin{aligned}\dot{\phi} &= -A(\alpha_0\delta + \alpha_1\delta^2 + \alpha_2\delta^3)\\ \dot{\delta} &\approx -B\left(\phi - \frac{1}{6}\phi^3 + \ldots\right),\end{aligned} \tag{6}$$

where $B = \mathrm{sgn}(\phi_*)B$, $\mathrm{sgn}(0) = 1$ and $\mathrm{sgn}(\pi) = -1$. We can now rewrite the above equations as

$$\ddot{\delta} - \zeta(\delta + \lambda\delta^2 + \mu\delta^3) = \epsilon F(\phi, \delta), \tag{7}$$

with $\zeta = AB\alpha_0$, $\lambda = \frac{\alpha_1}{\alpha_0}$, $\mu = \frac{\alpha_2}{\alpha_0}$ and treating the higher orders as a perturbation $\epsilon F(\phi, \delta) = -\frac{\zeta}{2}\phi^2(\delta + \lambda\delta^2 + \mu\delta^3)$.

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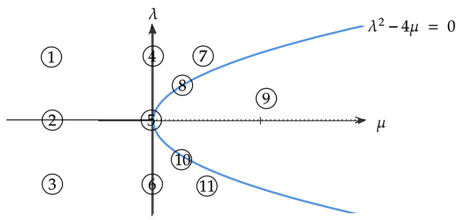

Figure 1: Bifurcation diagram for the remaining two parameters λ, μ with the regions of interest marked in numbers from 1-11.

UNPERTURBED ALPHA BUCKETS

As a first pass let us look at the unperturbed system $\epsilon = 0$ and examine

$$\ddot{\delta} - \zeta(\delta + \lambda\delta^2 + \mu\delta^3) = 0. \tag{8}$$

If $\zeta > 0$ we have the expansion around $\phi_* = 0$ and for $\zeta < 0$ we have the expansion around $\phi_* = \pi$. This is our first global bifurcation example. The system undergoes this bifurcation in 2 ways: either we change $\mathrm{sgn}(\zeta)$ by having a negative momentum compaction factor α_0 or reversing the RF-voltage (akin to introducing a π phase shift in the cavity). As can be easily seen from the above differential equation, we either have a harmonic or a hyperbolic solution. Hence, this global bifurcation will result in flipping between stable and unstable fixpoints in our phase plots.

The remaining three fixpoints of the system are

$$\begin{aligned} \delta_0 &= 0 \\ \delta_\pm &= \frac{-\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2\mu}, \quad \mu \neq 0 \\ \delta_\pm &= -\frac{1}{\lambda}, \quad \mu = 0. \end{aligned} \tag{9}$$

We can draw a 2D bifurcation diagram for the remaining two parameters λ, μ as depicted in Fig. 1. We have sketched the corresponding particle behaviour in longitudinal phase space for each region and show the global bifurcation $\zeta \to -\zeta$ by comparing Fig. 2 and Fig. 3.

We can now take a closer look at regions 1-11 in Figs. 1, 2 and 3. For region 5 we have effectively the equation of the harmonic oscillator, depending on the sign of ζ this will have stable orbits around 0 and unstable hyperbolic behaviour around π and vice-versa. For region 9 the only stable fixpoint is at $(0,0)$ or $(0,\pi)$ (again depending on $\mathrm{sgn}\,\zeta$) since the remaining fixpoints are imaginary.

The region pairs (1,3), (4,6), (7,11), and (8,10) reflect the symmetry of the bifurcation diagram in the λ-axis. By switching $\lambda \to -\lambda$ we effectively mirror the behaviour over the fixpoint (0,0) since the position of the stable and unstable fixpoints poses the same symmetry. This symmetry is completely seen in region 2. [1]

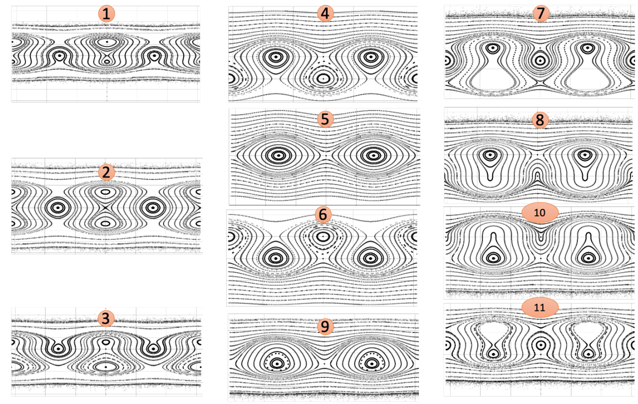

Figure 2: Schematics corresponding to the regions in diagram (Fig. 1). We have $\zeta > 0$. The x-axis corresponds to values $\phi \in [-2\pi, 2\pi]$ and the y-axis for $\delta \in [-4, 4]$ in relative units.

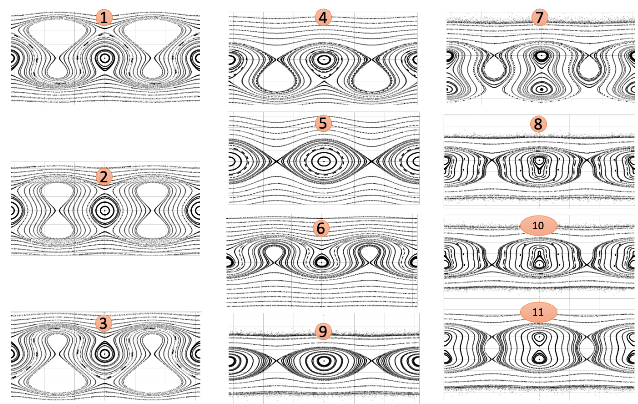

Figure 3: Schematics corresponding to the regions in diagram (Fig. 1). We have $\zeta < 0$. The x-axis corresponds to values $\phi \in [-2\pi, 2\pi]$ and the y-axis for $\delta \in [-4, 4]$ in relative units.

For regions 7 and 11, α_1^2 has to dominate over $4\alpha_0\alpha_2$, meaning the ratio $\frac{\alpha_0\alpha_2}{\alpha_1^2}$ cannot exceed 25 %.

Regions 8 and 10 demonstrate the limiting behaviour where two fixpoints merge together and become degenerate.

HOPF BIFURCATIONS

Let us now examine the perturbed equation

$$\ddot{\delta} - \zeta(\delta + \lambda\delta^2 + \mu\delta^3) = \epsilon F(\phi, \delta), \tag{10}$$

[1] Since sextupoles are the dominant element to tune α_1, if we set them in such a way that $\alpha_1 \approx 0$, we can expand the stable region by pushing the stable fixpoints further apart; their positions are then given by $\pm\frac{\sqrt{-\mu}}{\mu}$ and one must only be careful not to destroy the envelope region as discussed in the next section on Hopf bifurcations.

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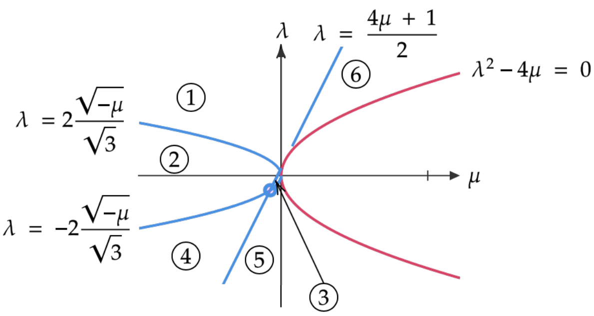

Figure 4: Hopf curves drawn into the bifurcation diagram Fig. 1. We have $\zeta > 0$.

The characteristic polynomial for eigenvalues l at any of the critical points is

$$l^2 + \epsilon \frac{\partial F}{\partial \phi} l + \text{const.} = 0. \tag{11}$$

We now have to examine the critical points for Hopf bifurcations (where the real part of the eigenvalues becomes zero) to find additional closed orbit solutions. We only need to focus on fixpoints where $\phi_* = 0$ due to the above mentioned global bifurcation $\zeta \to -\zeta$.
For α-buckets the above characteristic equation does not help us since

$$\frac{\partial F}{\partial \phi} = 0 \tag{12}$$

for any values of λ, μ and we cannot deduce any information from it. We have to resort to higher order expansions beyond linearization of the second order differential Eq. (10). The first non-trivial derivative yields more information. The fixpoint $\delta_* = 0$ is trivial. If we insert the fixpoints $\delta_\pm$ we get

$$\begin{aligned} \frac{\partial^3 F(\phi,\delta)}{\partial \phi^2 \partial \delta}\Big|_{(0,\delta_\pm)} &= -4\mu + 3\lambda^2 - 2\lambda \\ &\pm \sqrt{\lambda^2 - 4\mu} \mp 3\lambda\sqrt{\lambda^2 - 4\mu} \stackrel{!}{=} 0. \end{aligned} \tag{13}$$

With this we are able to amend our previous bifurcation diagram (Fig. 1) with the lines for Hopf bifurcations as depicted in Fig. 4. We have again sketched the corresponding particle behaviour for the regions of interest as can be seen in Fig. 5. We see that by crossing from region 1 → 2 our system obtains additional periodic solutions around the stable fixpoints $\delta_\pm$ creating an envelope. Such an occurrence may be used as a lifetime feedback "re-injecting" a particle from the lower into the upper bucket (for comp. [3, Fig. 5.14]) to exactly compensate the lifetime losses. Crossing the boundary line of region 6 reduces the existing periodic solutions to a negligible neighborhood of the fixpoint.

CONCLUSION

With this contribution we hope to have presented bifurcation theory as a powerful tool to be employed in the accelerator community to control non-linear beam dynamics. The fundamental shift from phase space to parameter space allows us to decide which particular non-linear effect we wish to employ in the accelerator and then find the corresponding hypersurface in parameter space where this is possible. Furthermore, we are also able to place stringent conditions between parameters as algebraic constraints which allows us to maximize or minimize effects. We have shown that creation of additional periodic solutions missed by leading order approximations is determined by the condition of Hopf bifurcations to occur. As an interesting extension we propose to find homoclinic bifurcations by constructing so called Poincaré mappings in the spirit of [5, chap. 4].

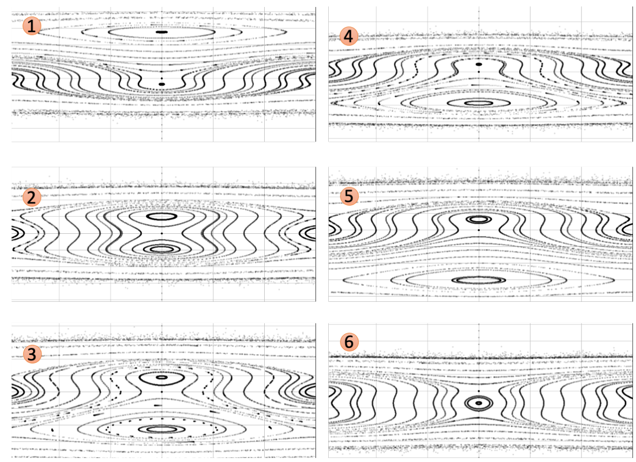

Figure 5: Creation or annihilation of periodic solutions depending on the Hopf region from Fig. 4. We have $\zeta > 0$ and the diagram is magnified only around the region of $\phi = 0$.

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