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LINEAR TRANSFER MATRIX OF A HALF SOLENOID*

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Abstract

Solenoid magnets can provide strong transverse focusing to electrons and ions with relatively small energies. For the ECR heavy-ion source, the ions are extracted at the central area of the solenoid, the beam is coupled at the exit of the source. The coupling caused by the solenoids can lead to the growth of projected transverse emittance, which has been widely studied with great interest. It is important to study the transfer matrix of a half solenoid to study the beam optics in an ECR souce, thus the property of the beam can be given. Based on the transfer matrix calculation, the summary of the linear transfer matrix of a half solenoid can be given. The beam optics in a half solenoid is studied.

INTRODUCTION

Solenoid magnets can provide strong transverse focusing to electrons and ions with relatively small energies. They have been used in the electron guns and the heavy ion sources [1–3]. the solenoids are also used to introduce transverse coupling [4]. The coupling caused by the solenoids can lead to the growth of projected transverse emittance, which has been widely studied with great interest.

In accelerator physics, the most commonly used solution in the form of transfer matrix for the calculation of the beam optics in solenoids can be found in [5] with hard-edge model. It is also reformed in [6]. To study the influence of the fringe field of the solenoids, delta fringe and linear fringe cases are considered [7, 8]. Also, an analytic approach for nonlinear beam dynamics in solenoids has been performed in [9].

For the ECR heavy-ion source, the ions are extracted at the central area of the solenoid, the beam is coupled at the exit of the source [3]. It is important to study the transfer matrix of a half solenoid to study the beam optics in an ECR souce, thus the property of the beam can be given.

In this paper, the summary of the linear transfer matrix of a half solenoid is given. The beam optics in a half solenoid is studied.

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BASIC DEFINITIONS

The beam matrix is used to describe the beam. The beam matrix is symmetric with ten independent variables,

$$\Sigma = \begin{pmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'x'\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle x'y\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'y'\rangle \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}. \tag{1}$$

The four-dimensional RMS emittance is $\varepsilon_{4D} = \sqrt{\det(\Sigma)}$, projected rms emittances in x plane is $\varepsilon_x = \sqrt{\det(\sigma_{xx})}$, and projected rms emittance in y plane is $\varepsilon_y = \sqrt{\det(\sigma_{yy})}$.

The beam matrix at s_2 can be calculated by transfer matrix R from s_1 to s_2 and the beam matrix at s_1,

$$\Sigma(s_2) = R\Sigma(s_1)R^T. \tag{2}$$

If R obeys the symplecticity condition [10, 11], ε_{4D} is conserved. The transfer matrices of most linear elements obey the symplecticity condition. It is important to study the trans-fer matrix for the beam dynmaics study.

APPROACH TO TRANSFER MATRIX OF A HALF SOLENOID

To study the beam optics in the ion source, the transfer matrix of a half exiting solenoid is needed.

Traditional Formulation

The transfer matrix for a hard-edge solenoid can be solved as [5]

$$M(L^+|0^-) = \begin{pmatrix} \cos^2\Theta & \frac{1}{2k}\sin 2\Theta & \frac{1}{2}\sin 2\Theta & \frac{1}{k}\sin^2\Theta \\ -\frac{k}{2}\sin 2\Theta & \cos^2\Theta & -k\sin^2\Theta & \frac{1}{2}\sin 2\Theta \\ -\frac{1}{2}\sin 2\Theta & -\frac{1}{k}\sin^2\Theta & \cos^2\Theta & \frac{1}{2k}\sin 2\Theta \\ k\sin^2\Theta & -\frac{1}{2}\sin 2\Theta & -\frac{k}{2}\sin 2\Theta & \cos^2\Theta \end{pmatrix}, \tag{3}$$

where $\Theta = kL$, $k = B_{z0}/2[B\rho]$, $[B\rho] = mc\beta\gamma/q$, and

$$B_{z0} = \frac{\int_{-\infty}^{\infty} dz B_z^2(z)}{\int_{-\infty}^{\infty} dz B_z(z)}, L = \frac{\left[\int_{-\infty}^{\infty} dz B_z(z)\right]^2}{\int_{-\infty}^{\infty} dz B_z^2(z)}. \tag{4}$$

The transfer matrix can be resolved as

$$M(L^+|0^-) = K_{\text{out}} M_s K_{\text{in}}, \tag{5}$$

where

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$$K_{\text{out}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix}, K_{\text{in}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix},$$
$$M_s = \begin{pmatrix} 1 & \frac{1}{2k}\sin 2\Theta & 0 & \frac{1}{k}\sin^2\Theta \\ 0 & \cos 2\Theta & 0 & \sin 2\Theta \\ 0 & -\frac{1}{k}\sin^2\Theta & 1 & \frac{1}{2k}\sin 2\Theta \\ 0 & -\sin 2\Theta & 0 & \cos 2\Theta \end{pmatrix}, \tag{6}$$

Thus, the transfer matrix of a half exiting solenoid can be solved as

$$M(L^+|L/2) = K_{\text{out}} M_{\text{half}} \ . \tag{7}$$

Besides, the transfer matrix of a half exiting solenoid can ce solved from the half entering solenoid. The transfer matrix of the half entering solenoid is

$$M(L/2|0^-) = \begin{pmatrix} \cos^2\frac{\Theta}{2} & \frac{1}{2k}\sin\Theta & \frac{1}{2}\sin\Theta & \frac{1}{k}\sin^2\frac{\Theta}{2} \\ -\frac{k}{2}sin\Theta & \cos\Theta & k\cos\Theta & \sin\Theta \\ -\frac{1}{2}\sin\Theta & -\frac{1}{k}\sin^2\frac{\Theta}{2} & \cos^2\frac{\Theta}{2} & \frac{1}{2k}\sin\Theta \\ -k\cos\Theta & -\sin\Theta & -\frac{k}{2}sin\Theta & \cos\Theta \end{pmatrix}. \tag{8}$$

As the magnetic field is symmetric, we can immediately write out the transfer matrix of the half exiting solenoid using the property of T-symmetry [9]:

$$M(L^+|L/2) = JM(L/2|0^-)J, \tag{9a}$$

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9b}$$

thus the transfer matrix of the half exiting solenoid can be obtained,

$$M(L^+|L/2) = \begin{pmatrix} 1 & \frac{1}{2k}\sin\Theta & 0 & \frac{1}{k}\sin^2\frac{\Theta}{2} \\ 0 & \cos^2\Theta & -k & \frac{1}{2}\sin\Theta \\ 0 & -\frac{1}{k}\sin^2\frac{\Theta}{2} & 1 & \frac{1}{2k}\sin\Theta \\ k & -\frac{1}{2}\sin\Theta & 0 & \cos\Theta \end{pmatrix}, \tag{10}$$

which is same as Eq. (7).

Reformulation

The formula in [5,6] is used to claculate the transfer matrix from z_0 to z_1 in the solenoid:

$$M(z_1|z_0) = R_1 \cdot L_1 \cdot R_0^{-1} = M_r(z_1|z_0) \cdot M_L(z_1|z_0) \cdot M_r^{-1}(z_0|z_0), \tag{11}$$

$R_i = M_r$ is the transform between the laboratory and Larmor frames at coordinate z_i, and $L_i = M_L$ is the Larmor-frame transfer matrix from z_{i-1} to z_i, $\Delta z = z_i - z_{i-1}$. The expressions are

$$R_i = \begin{pmatrix} P_i & -Q_i \\ Q_i & P_i \end{pmatrix}, L_i = \begin{pmatrix} F_i & 0 \\ 0 & F_i \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} P_i &= \begin{pmatrix} \cos\theta_{ri} & 0 \\ -\theta'_{ri}\sin\theta_{ri} & \cos\theta_{ri} \end{pmatrix}, Q_i = \begin{pmatrix} \sin\theta_{ri} & 0 \\ \theta'_{ri}\cos\theta_{ri} & \sin\theta_{ri} \end{pmatrix}, \\ F_i &= \begin{pmatrix} \cos\theta_{Li} & \sin\theta_{Li}/k_{Li} \\ -k_{Li}\sin\theta_{Li} & \cos\theta_{Li} \end{pmatrix}, \\ \theta_{ri} &= -\int_z^{z_i} dz B_z(z)/2[B\rho], \\ \theta'_{ri} &= -k_{ri} = -B_z(z_i)/2[B\rho], \\ k_{Li} &= \frac{1}{2[B\rho]}\sqrt{\frac{1}{\Delta z}\int_{z_{i-1}}^{z_i} dz B_z^2(z)}, \\ \theta_{Li} &= k_{Li}\Delta z. \end{aligned} \tag{13}$$

Thus the transfer matrix from z_0 to z_n in the solenoid can be obtained based on slicing model:

$$\begin{aligned} M(z_n|z_0) &= \prod_{i=1}^{n} R_i \cdot L_i \cdot R_{i-1}^{-1} = R_n \cdot \prod_{i=1}^{n} L_i \cdot R_0^{-1} \\ &= \begin{pmatrix} P_n \prod F_i P_0 + Q_n \prod F_i Q_0 & P_n \prod F_i Q_0 - Q_n \prod F_i P_0 \\ Q_n \prod F_i P_0 - P_n \prod F_i Q_0 & Q_n \prod F_i Q_0 + P_n \prod F_i P_0 \end{pmatrix}, \end{aligned} \tag{14}$$

To solve the transfer matrix of the exiting half solenoid, $B_z(L) = 0$. it can be calculated that

$$\begin{aligned} M(L^+|L/2) &= R_L \cdot L \cdot R_{L/2}^{-1} \ , \\ R_{L/2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_r(L/2) & 0 \\ 0 & 0 & 1 & 0 \\ -k_r(L/2) & 0 & 0 & 1 \end{pmatrix}, \\ R_L &= \begin{pmatrix} \cos\theta_{rn} & 0 & -\sin\theta_{rn} & 0 \\ -k_T\sin\theta_{rn} & \cos\theta_{rn} & -k_T\cos\theta_{rn} & -\sin\theta_{rn} \\ \sin\theta_{rn} & 0 & \cos\theta_{rn} & 0 \\ k_T\cos\theta_{rn} & \sin\theta_{rn} & -k_T\sin\theta_{rn} & \cos\theta_{rn} \end{pmatrix}, \\ L &= \begin{pmatrix} \cos\theta_L & \sin\theta_L/k_L & 0 & 0 \\ -k_L\sin\theta_L & \cos\theta_L & 0 & 0 \\ 0 & 0 & \cos\theta_L & \sin\theta_L/k_L \\ 0 & 0 & -k_L\sin\theta_L & \cos\theta_L \end{pmatrix}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} k_r(L/2) &= B_z(L/2)/2[B\rho], k_T = B_z(L/2)/6[B\rho], \\ k_L &= \frac{1}{2[B\rho]}\sqrt{\frac{1}{L/2}\int_{L/2}^{L} dz B_z^2(z)}, \theta_L = k_L L/2, \\ \theta_{rn} &= -\int_{L/2}^{L} dz B_z(z)/2[B\rho]. \end{aligned} \tag{16}$$

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The length of the solenoid can be calculated using linear fringe field (see Fig. 1):

$$C = \frac{\int_{L/2}^{\infty} dz B_z(z)}{B_{\max}}, D = \frac{\int_{L/2}^{\infty} dz B_z^2(z)}{B_{\max}^2}, \quad (17)$$
$$L_1 = 2C, L/2 = 4C - 3D.$$

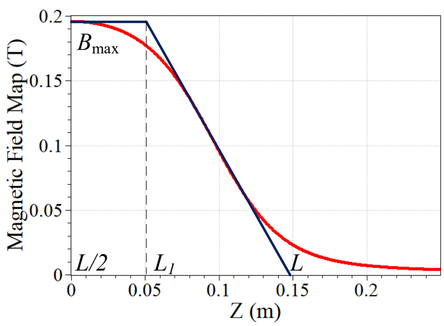

Figure 1: Real fringe field and linear fringe field of a half solenoid.

PROPERTY OF THE TRANSFER MATRIX

Traditional Formulation

It can be calculated that the transfer matrix of the whole solenoid obeys the symplecticity condition: $M(L^+|0^-)^T SM(L^+|0^-) = S$.

With the traditional formulation, we can calculate that

$$M(L^+|L/2)^T SM(L^+|L/2) = \begin{pmatrix} 0 & 1 & -2k & 0 \\ -1 & 0 & 0 & 0 \\ 2k & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (18)$$

which implies that the transfer matrix of the half exiting solenoid is non-symplectic. ε_{4D} is no longer conserved.

BEAM PROPERTY

Traditional Formulation

Assume the beam matrix at the center of the solenoid

$$\Sigma_{\text{in}} = \varepsilon \begin{pmatrix} \beta & -\alpha & 0 & 0 \\ -\alpha & \gamma & 0 & 0 \\ 0 & 0 & \beta & -\alpha \\ 0 & 0 & -\alpha & \gamma \end{pmatrix}, \quad (19)$$

where α, β and γ are Twiss parameters and $\beta\gamma - \alpha^2 = 1$. The beam matrix at the exit of the half solenoid can be obtained using the traditional formulation:

$$\Sigma_{\text{out}} = M(L^+|L/2)\Sigma_{\text{in}} M(L^+|L/2)^T, \quad (20)$$

thus

$$\Sigma_{\text{out}} = \varepsilon \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_{yy} \end{pmatrix}, \quad (21)$$

where

$$\Sigma_{xx} = \Sigma_{yy} = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}, \Sigma_{xy} = \begin{pmatrix} 0 & \beta k \\ -\beta k & 0 \end{pmatrix},$$
$$r_{11} = \frac{1}{2k^2}(2\beta k^2 - 2\alpha k \sin\Theta + \gamma - \gamma\cos\frac{\Theta}{2}),$$
$$r_{12} = \frac{1}{2k}(\gamma\sin\Theta - 2\alpha\cos\Theta), \quad (22)$$
$$r_{22} = \beta k^2 + \alpha k \sin\Theta + \gamma\cos^2\frac{\Theta}{2},$$

ε_x and ε_y at the exit of the half solenoid are $\varepsilon\sqrt{1+\beta^2 k^2}$. It can be calculated that ε_{4D} is conserved if the beam is decoupled with same parameters in $x-x'$ and $y-y'$ planes at the center of the solenoid. It suggests that the projected rms emittancs will be increased at the exit of the half solenoid. The beam is transversely-coupled.

Comparison

We consider the simulation of the phase spaces at the exit of an exiting half solenoid. The real field is used (see Fig. 1). The input beam is $^{16}O^{5+}$ with kinetic energy of 75 keV. The normalized rms emittance is 0.1 π mm·mrad with $\alpha = 0$, $\beta = 0.4$ mm/mrad. The comparison of the traditional formulation, reformulation and the real field results is given in Fig. 2.

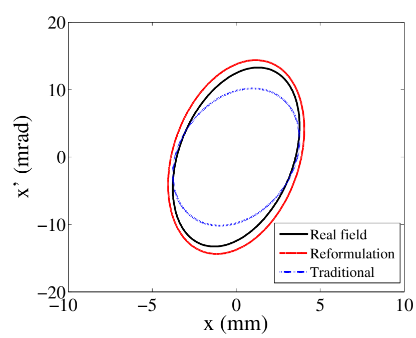

Figure 2: Comparison of the phase spaces at the exit of the exiting half solenoid.

CONCLUSION

This paper presents a summary of the linear transfer matrix of a half solenoid. Also, beam optics study of a half solenoid is given, reveals the emittance growth and beam coupling due to the half solenoid. The result of reformulation is closer to the one of real field.

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