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EIGENMODE DECOMPOSITION FOR FREE-ELECTRON LASERS USING BAYESIAN ANALYSIS*

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Abstract

Laser beams from an optical cavity, such as free-electron laser (FEL) resonators, are typically a mixture of the cavity's eigenmodes, such as the Hermite-Gaussian (HG) modes or Laguerre-Gaussian (LG) modes. Robust evaluation of the eigenmode spectrum of a multimode laser beam has various applications in laser development, research, and utilization. In this work, a general eigenmode decomposition method for a multimode laser beam has been developed based on Bayesian analysis. This problem is transformed into a linear system and then solved using a Gaussian probabilistic model. Using Bayesian analysis, prior knowledge about the mode content is further incorporated into the solution to improve the results for laser beams contaminated with complex disturbances. The decomposition of the beam image from the incoherent intensity addition of HG modes is discussed with different types of noise or disturbances. The simulation results have been used to show the robustness of this method. This method can be straightforwardly extended into other cases such as the wavefront decomposition into the coherent superposition of HG and LG modes.

INTRODUCTION

The output beam from an FEL resonator can be composed of multiple transverse eigenmodes, such as the Hermite-Gaussian (HG) eigenmodes in the rectangular coordinates or Laguerre-Gaussian (LG) eigenmodes in cylindrical coordinates [1]. If the transverse eigenmode components with different mode orders have slightly different frequencies, the measured intensity of such a multimode beam can be treated as a direct sum of the intensity of underlying eigenmodes. This is because the camera exposure time is usually long enough so that coherent interference among the modes of slightly different frequencies is washed out, where such superposition can be referred to as "incoherent."

To optimize the operation of a multimode FEL beam, it can be useful to obtain the knowledge of the mode content based on measured beam images. Traditionally, mode decomposition of coherent and incoherent beams can be done using various techniques when the eigenmodes are orthogonal [2–7]. When the mode basis is not orthogonal, a transformation method from the non-orthogonal basis to an orthogonal one can be used [8–10]. To accurately obtain eigenmode decomposition results, however, the computation cost is usually high for the transformation method when applied to high-resolution images. Recently, phase retrieval techniques that are not wavelength limited have been explored [11–15]. With the retrieved phase, eigenmode decomposition can be done using a simple inner product with the orthogonal eigenmodes. However, this method may result in a relatively large residue using low-resolution images or a high computational cost using high-resolution images when a large number of eigenmodes are involved.

To overcome these shortcomings, an eigenmode decomposition technique that is suitable for both incoherent and coherent multimode FEL beams in a wide spectral range and also has a low computational cost is desired. This can be developed using machine learning techniques, which have been rapidly advanced with applications in many fields including the optical field [16–18]. In this work, we present a new eigenmode decomposition method using Bayesian analysis. Various simulation results have proven the feasibility of this method, which can be further extended to other situations [19].

METHOD

The intensity of a multimode FEL beam by the incoherent superposition of HG modes can be written as [19]

$$Y(x,y) = \sum_{p,q} \alpha_{p,q}\phi_p^2(x)\phi_q^2(y) = \vec{f}(x)X\vec{g}(y)^T, \quad (1)$$

where $\vec{f}(x)$ and $\vec{g}(y)$ are HG basis vectors, and the desired modal weights are tabulated in the weight matrix

$$X = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m,0} & \alpha_{m,1} & \cdots & \alpha_{m,n} \end{bmatrix}. \quad (2)$$

Usually, the noise in the intensity measurement from a normal-operating digital camera can be modeled a Gaussian distribution, and different pixels can be considered to be almost independent. Therefore we assume that for an intensity measurement Y with $M \times N$ pixels, the measured intensity $Y_{i,j}$ at any location (x_i, y_j) is normally distributed around $\vec{f}(x_i)X\vec{g}(y_j)^T$ with a common variance σ^2. With this assumption, the likelihood of the measured intensity Y given a weight matrix X is

$$p(Y|X) = \prod_{i,j} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left[Y_{i,j} - \vec{f}(x_i)X\vec{g}(y_j)^T\right]^2}{2\sigma^2} \right). \quad (3)$$

The optimal weight matrix $\hat{X}$ can be obtained by maximizing the likelihood $p(Y|X)$, or equivalently minimizing

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$-\log p(Y|X)$, with

$$-\log p(Y|X) \propto \sum_{i,j} \left[Y_{i,j} - \vec{f}(x_i) X \vec{g}(y_j)^T\right]^2 = \|Q\|_F^2, \quad (4)$$

where the Frobenius norm $\|Q\|_F^2 = Tr(Q^T Q)$, $Q = Y - P_x X P_y^T$, and matrix P_x is defined as

$$(P_x)_{i,p} = \phi_p^2(x_i) = \left[\vec{f}(x_1)^T, \vec{f}(x_2)^T, \cdots, \vec{f}(x_M)^T\right]^T. \quad (5)$$

Using the matrix calculus, it can be found that

$$\hat{X} = (P_x^T P_x)^{-1} (P_x^T Y P_y)(P_y^T P_y)^{-1}, \quad (6)$$

which is a unique solution to the eigenmode decomposition in the HG mode basis because matrix $P_x^T P_x$ (or $P_y^T P_y$) is positive definite. Using this optimal solution $\hat{X}$, the variance $\hat{\sigma}^2$ is found to be

$$\hat{\sigma}^2 = \frac{1}{MN} \|Y - P_x \hat{X} P_y^T\|_F^2. \quad (7)$$

From the FEL operation or the observation of the beam image, certain knowledge such as the dominant mode is available, which can be built into a prior weight matrix X_0 and fed into the optimization. Assuming modal weights X_{ij} are independently and normally distributed around prior $(X_0)_{ij}$ with a common variance τ^2, the prior probability about X can be written as

$$p(X) \propto \prod_{i,j} \exp\left(-\frac{\left[X_{i,j} - (X_0)_{i,j}\right]^2}{2\tau^2}\right). \quad (8)$$

Using Bayes' theorem, the posterior probability is $p(X|Y) \propto p(Y|X)p(X)$. By maximizing $p(X|Y)$, or equivalently, minimizing

$$-\log p(X|Y) \propto \|Y - P_x X P_y^T\|_F^2 + \lambda \|X - X_0\|_F^2, \quad (9)$$

with $\lambda = \sigma^2/\tau^2$ a free parameter representing the regularization strength, the optimal $\hat{X}$ can be found from solving the Sylvester equation

$$AX + XB = C, \quad (10)$$

with $A = P_x^T P_x$, $B = \lambda (P_y^T P_y)^{-1}$, and $C = (P_x^T Y P_y + \lambda X_0)(P_y^T P_y)^{-1}$. This equation can be efficiently and uniquely solved when A and $-B$ do not share common eigenvalues [20]. For the mode decomposition in HG mode basis, because $P_x^T P_x$ and $P_y^T P_y$ are positive definite and $\lambda \geq 0$, the solution in Eq. (10) is unique.

SIMULATION RESULTS

Simulations were performed to validate this method using laser wavelength $\lambda = 450$ nm and Rayleigh range $z_R = 10$ cm. This method can be used at any longitudinal location. As a demonstration, the intensity was computed at the waist location ($z = 0$) in a 1×1 mm^2 region with pixel numbers $M = N = 301$. The mode decomposition was done with the highest mode numbers $m = n = 5$.

Figure 1 illustrates mode decomposition results using the method without regularization using test beams of different levels of Gaussian noise. The test beam is generated by mixing an ideal beam, with three modes $HG_{0,0}$ (65%), $HG_{0,1}$ (20%), and $HG_{2,0}$ (15%), and Gaussian noise of different noise level σ_n, which is normalized to the maximum intensity value of the ideal beam. Eqs. (6) and (7) are applied to obtain the modal weights and estimated variance. The retrieved modal weights are used to reconstruct the output intensity. As shown in Fig. 1, even with a large σ_n up to 0.2, where noise overwhelms the lowest intensity mode(s), the algorithm still correctly recovers the ideal beam intensity, and the estimated $\hat{\sigma}$ is in excellent agreement with the input noise level σ_n.

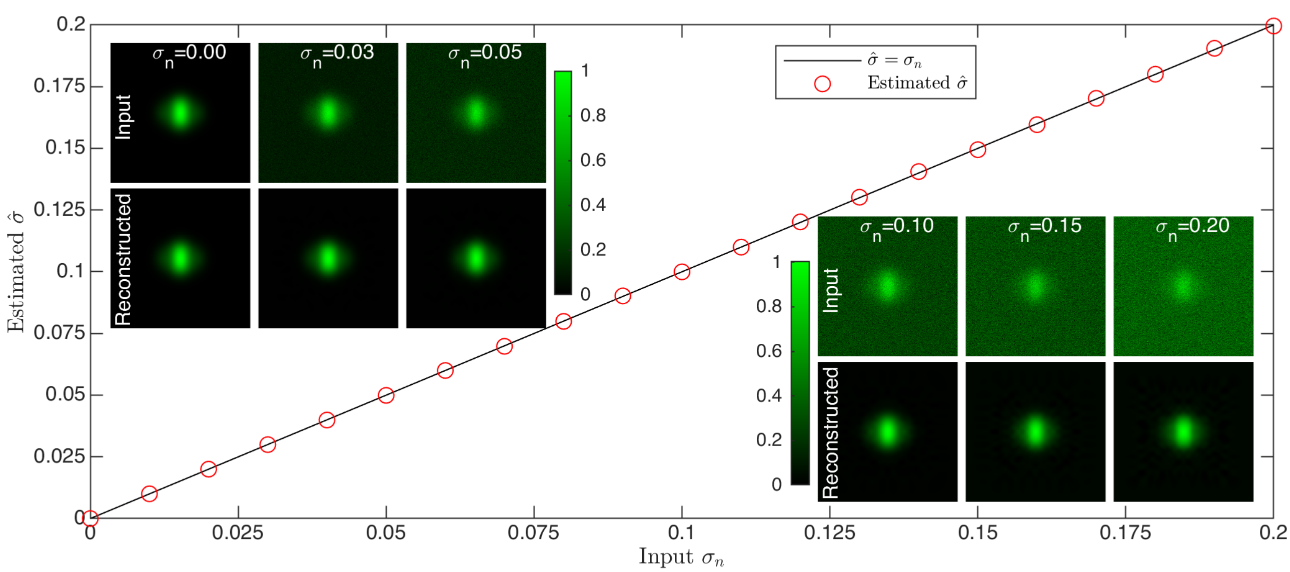

Figure 1: Mode decomposition and image reconstruction results. Gaussian noises with different levels (input σ_n) are added to the same ideal beam (65% $HG_{0,0}$, 20% $HG_{0,1}$, and 15% $HG_{2,0}$) to obtain the input images. Using these images, the modal weights and $\hat{\sigma}^2$ are retrieved. The insets show the input and reconstructed images at different σ_n values.

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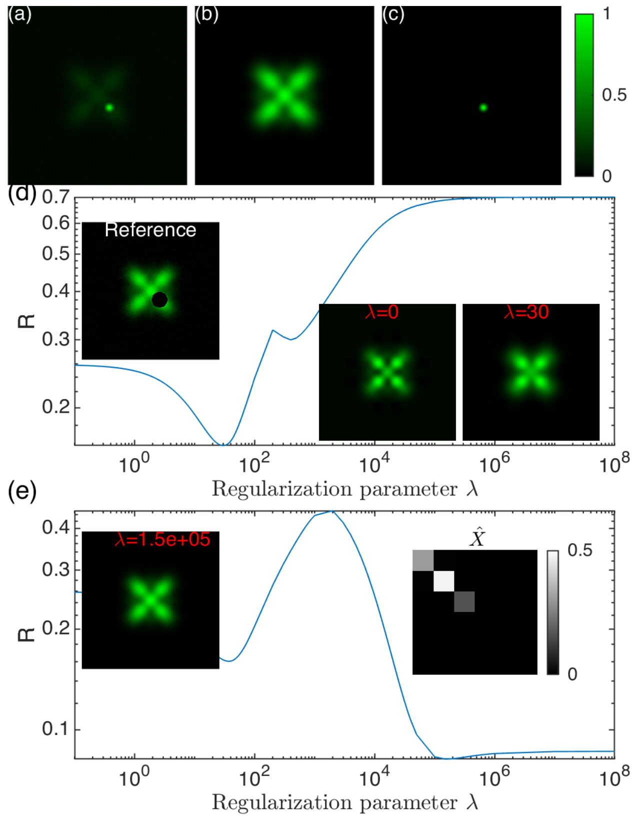

Figure 2: Modal analysis with different values of regularization parameter λ. Test beam (a) is the sum of an ideal beam (b), a diffraction pattern (c), and Gaussian noise with $\sigma_n = 0.05$; (d) Intensity difference R between the reconstructed image and reference image (top-left inset) as a function of λ. The two bottom-right insets are the reconstructed images with zero regularization ($\lambda = 0$) and optimal regularization ($\lambda = 30$), respectively. (e) Intensity difference R after the second iteration. The top-left inset is the reconstructed image with optimal regularization ($\lambda = 1.5 \times 10^5$). The top-right inset shows the corresponding reconstructed weight matrix $\hat{X}$.

For images with symmetric noise such as Gaussian noise, the simulation results without regularization are already good enough. However, with some complex disturbances such as interference or diffraction, the inclusion of prior knowledge about dominant modes can result in a better solution. For example, complex "noise" from the diffraction of a dust particle can usually be observed on intensity measurements. To resolve this type of noise, regularization is used with an image contaminated with diffraction noise [Fig. 2(a)], which is the sum of the ideal beam Fig. 2(b), Gaussian noise with $\sigma_n = 0.05$, and diffraction patterns Fig. 2(c). The ideal beam is composed of 30% $HG_{0,0}$, 50% $HG_{1,1}$, and 20% $HG_{2,2}$. To use the regularized solution Eq. (10), prior knowledge X_0 with one nonzero element (its value set to unity) corresponding to the $HG_{0,0}$ mode is constructed. To choose the optimal regularization parameter λ, a reference image can be constructed by excluding regions of the image contaminated by diffraction as shown in the top-left inset of Fig. 2(d), and the relative difference R is calculated at multiple λ values. As shown in Fig. 2(d), the relative difference is minimized at $\lambda = 30$, resulting in a reconstructed image which is much closer to the ideal image than the one obtained without regularization ($\lambda = 0$). Using the optimal weight matrix as a prior, a second iteration can be done, producing a further improved reconstructed image shown in Fig. 2(e).

CONCLUSION

In this paper, we have presented a deterministic eigenmode decomposition method for FEL transverse modes based on Bayesian analysis. This method is shown, by various simulation results, to be able to reconstruct the beam intensity with Gaussian noise of high noise level. When complex disturbances are present in the beam image, prior knowledge can be used to greatly improve the reconstruction results. While this method is introduced for the case in which the beam image is an incoherent intensity superposition of HG modes, it can be easily extended to other situations in which the beam image represents a coherent superposition of HG or LG modes using the generalized formulation in Ref. [19].

REFERENCES

[1] A. E. Siegman, *Lasers*. Mill Valley, CA: University Science Books, 1986.

[2] E. Tervonen, J. Turunen, and A. T. Friberg, "Transverse laser-mode structure determination from spatial coherence measurements: Experimental results", *Applied Physics B*, vol. 49, no. 5, pp. 409–414, 1989. doi:10.1007/BF00325341

[3] A. Cutolo, T. Isernia, I. Izzo, R. Pierri, and L. Zeni, "Transverse mode analysis of a laser beam by near-and far-field intensity measurements", *Applied Optics*, vol. 34, no. 34, pp. 7974–7978, 1995. doi:10.1364/AO.34.007974

[4] C. M. Warnky, B. L. Anderson, and C. A. Klein, "Determining spatial modes of lasers with spatial coherence measurements", *Applied Optics*, vol. 39, no. 33, pp. 6109–6117, 2000. doi:10.1364/AO.39.006109

[5] X. Xue, H. Wei, and A. G. Kirk, "Intensity-based modal decomposition of optical beams in terms of Hermite–Gaussian functions", *Journal of the Optical Society of America A*, vol. 17, no. 6, pp. 1086–1091, 2000. doi:10.1364/JOSAA.17.001086

[6] F. Gori, M. Santarsiero, R. Simon, G. Piquero, R. Borghi, and G. Guattari, "Coherent-mode decomposition of partially polarized, partially coherent sources", *Journal of the Optical Society of America A*, vol. 20, no. 1, pp. 78–84, 2003. doi:10.1364/JOSAA.20.000078

[7] Y. Li, J. Guo, L. Liu, T. Wang, and J. Shao, "Linear equations method for modal decomposition using intensity information", *Journal of the Optical Society of America A*, vol. 30, no. 12, pp. 2502–2509, 2013. doi:10.1364/JOSAA.30.002502

[8] F. Gori, M. Santarsiero, R. Borghi, and G. Guattari, "Intensity-based modal analysis of partially coherent beams with Hermite–Gaussian modes", *Optics Letters*, vol. 23, no. 13, pp. 989–991, 1998. doi:10.1364/OL.23.000989

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[9] M. Santarsiero, F. Gori, R. Borghi, and G. Guattari, "Evaluation of the modal structure of light beams composed of incoherent mixtures of Hermite–Gaussian modes", *Applied Optics*, vol. 38, no. 25, pp. 5272–5281, 1999. doi:10.1364/AO.38.005272

[10] R. Borghi, G. Piquero, and M. Santarsiero, "Use of biorthogonal functions for the modal decomposition of multimode beams", *Optics Communications*, vol. 194, no. 4-6, pp. 235–242, 2001. doi:10.1016/S0030-4018(01)01187-7

[11] L. J. Allen and M. P. Oxley, "Phase retrieval from series of images obtained by defocus variation", *Optics Communications*, vol. 199, no. 1-4, pp. 65–75, 2001. doi:10.1016/S0030-4018(01)01556-5

[12] J. Primot, "Theoretical description of Shack–Hartmann wavefront sensor", *Optics Communications*, vol. 222, no. 1-6, pp. 81–92, 2003. doi:10.1016/S0030-4018(03)01565-7

[13] Y. Carmon and E. N. Ribak, "Phase retrieval by demodulation of a Hartmann–Shack sensor", *Optics Communications*, vol. 215, no. 4-6, pp. 285–288, 2003. doi:10.1016/S0030-4018(02)02254-X

[14] H. Huang *et al.*, "Phase-shift interference-based wavefront characterization for orbital angular momentum modes", *Optics Letters*, vol. 38, no. 13, pp. 2348–2350, 2013. doi:10.1364/OL.38.002348

[15] P. Liu, J. Yan, H. Hao, and Y. K. Wu, "Phase retrieval for short wavelength orbital angular momentum beams using knife-edge diffraction", *Optics Communications*, vol. 474, p. 126 077, 2020. doi:10.1016/j.optcom.2020.126077

[16] P. A. Bakut, V. E. Kirakosyants, V. A. Loginov, C. J. Solomon, and J. C. Dainty, "Optimal wavefront reconstruction from a Shack-Hartmann sensor by use of a Bayesian algorithm", *Optics Communications*, vol. 109, no. 1-2, pp. 10–15, 1994. doi:10.1016/0030-4018(94)90730-7

[17] T. Doster and A. T. Watnik, "Machine learning approach to OAM beam demultiplexing via convolutional neural networks", *Applied Optics*, vol. 56, no. 12, pp. 3386–3396, 2017. doi:10.1364/AO.56.003386

[18] S. Lohani and R. T. Glasser, "Turbulence correction with artificial neural networks", *Optics Letters*, vol. 43, no. 11, pp. 2611–2614, 2018. doi:10.1364/OL.43.002611

[19] P. Liu, J. Yan, W. Li, and Y. K. Wu, "Transverse mode analysis for free-space laser beams using bayesian analysis", *Applied Optics*, vol. 60, no. 12, pp. 3344–3352, 2021. doi:10.1364/AO.420217

[20] G. Golub, S. Nash, and C. Van Loan, "A Hessenberg-Schur method for the problem $AX + XB = C$", *IEEE Transactions on Automatic Control*, vol. 24, no. 6, pp. 909–913, 1979. doi:10.1109/TAC.1979.1102170