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AN EMPIRICALLY-DERIVED ABCD MATRIX FOR TRANSVERSE DYNAMICS STUDIES IN SEEDED FREE-ELECTRON LASERS

R. R. Robles[1]*, G. Marcus, Z. Huang[1], SLAC National Accelerator Laboratory
[1]also at Department of Applied Physics, Stanford University

Abstract

We present a simple empirical method for deriving an ABCD matrix for studying the transverse dynamics of the radiation field in seeded, high-gain free-electron lasers before saturation. In spite of the inherently nonlinear nature of FEL optical guiding, the ABCD matrix we find is able to predict the evolution of the FEL mode size and centroid to a high degree of accuracy across a large range of input mode characteristics. This scheme enables extremely fast simulation of transverse dynamics, which in turn greatly simplifies numerical studies of seeded FEL systems. Of particular interest in that regard is the x-ray regenerative amplifier free-electron laser, in which the x-ray beam propagates through an optical cavity many hundreds of times, thereby making traditional simulation methods cumbersome and time consuming.

INTRODUCTION

Free-electron lasers owe their exceptional transverse coherence to the phenomenon of optical guiding by which the electron beam focuses the radiation by both gain and refractive guiding [1]. This process ensures that the FEL output is dominated by a well-focused, roughly gaussian transverse mode with a size comparable to that of the electron beam itself [2, 3]. Since the electron beam transverse distribution is typically well-approximated as gaussian or parabolic, the transverse dynamics of the system can be modeled with high accuracy by treating the electron beam as an optical fiber with a complex quadratic refractive index [1, 4–6] which actively changes with the radiation beam [7]. This simple model for FEL dynamics greatly enhances the speed at which an FEL's properties can be simulated, and provides an intuitive picture of the guiding process. This analogy can be taken one step further, as it is well-known that quadratic gradient index media may be treated using the ABCD matrix formalism for even faster tracking. The analogy between the FEL gain medium and a quadratic optical fiber implies that the FEL dynamics may also be well-approximated by an effective ABCD matrix, which would even further enhance the speed of simulation studies.

In this paper we present an empirical model for deriving the effective ABCD matrix describing the transverse dynamics of an FEL. We will do so using the refractive index model presented in Ref. [7], which means that we inherit the assumptions of that approach. In particular, we will consider a single-frequency, seeded FEL with a radiation beam modelled as a perfect gaussian, and the analysis does not capture the effects of saturation.

* riverr@stanford.edu

These restrictions, although notable, still enable the approach to find a very natural use case in the x-ray regenerative amplifier (XRAFEL) [8, 9], in which the radiation is circulated around an optical cavity many times while it gets re-amplified with high gain by fresh electron bunches. Studying the dynamics of XRAFELs is extremely time consuming as a result of the many FEL simulations required to properly characterize it, so the ability to perform these studies using a matrix formalism is highly attractive. To highlight the particular utility of this method in understanding the impact of FEL guiding on XRAFEL alignment tolerances, we show that the ABCD matrix can be used to track both the radiation mode size as well as the radiation centroid in the case that the seed radiation has been perturbed off-axis by some error in the larger system.

EMPIRICAL METHOD

We will use the same formalism as was employed in Ref. [7]. As such, the radiation field is modelled as an ideal gaussian beam of the form

$$E(x, y, z) = f(z) \exp\left[-\frac{i}{2}\left(Q_x(z)(x - x_0(z))^2 + Q_y(z)y^2\right)\right]. \tag{1}$$

where the parameters $f(z)$, $Q_{x/y}(z)$, and $x_0(z)$ are complex and describe the field amplitude, mode size and divergence, and centroid offset and angle, respectively. As we will attempt to treat the FEL as an ABCD matrix element, we must expand this formalism to understand how a gaussian field transforms through an ABCD matrix. This can be done straightforwardly using the Huygens integral (see, for example, [10])

$$E(x_f, y_f) \propto \int K_x(x_i, x_f) K_y(y_i, y_f) E(x_i, y_i) dx_i dy_i, \tag{2}$$

where the integral kernels have the form

$$K_x(x_i, x_f) = \sqrt{\frac{i}{B\lambda_r}} \exp\left[-\frac{i\pi}{B\lambda_r}(Ax_i^2 - 2x_i x_f + Dx_f^2)\right], \tag{3}$$

with a similar expression in y. For our purposes the matrix values are complex. As our primary goal is to model the transverse FEL dynamics, we will not worry about the change in the field amplitude. For the gaussian beam these integrals result in another gaussian with mode parameters transformed according to

$$Q_{x,y} \to k_r \frac{k_r C + DQ_{x,y}}{k_r A + BQ_{x,y}}, \qquad x_0 \to \frac{Q_x x_0}{k_r C + DQ_x}. \tag{4}$$

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In principle, one can always simply define matrix elements such that the FEL behaves in this way, however in general the elements A, B, C, and D will vary with the input Q and x_0. This is to be expected - the FEL electron beam thought of as a focusing medium is naturally a highly non-linear one, since it only obtains these focusing properties through its interaction with the radiation beam. Nonetheless, we may make two approximations which we have justified via numerical studies. The first of these is that the fundamental focusing properties of the FEL do not change when the radiation beam is initialized off-axis, at least to lowest-order. An example of the impact of the input seed offset on the output radiation size is shown in Fig. 1 which makes use of the parameter set studied in the next section. It is clear from this figure that the seed offset has a second-order impact on the focusing properties, and that this is enhanced when the offset is comparable to the initial fwhm. As a result, as long as the initial radiation offset is not too large relative to its transverse size, we may assume that the ABCD elements do not depend on the initial value of x_0. This assumption implies that we may find the desired ABCD matrix for some particular initial value of x_0 and extrapolate this matrix to other initial values of x_0.

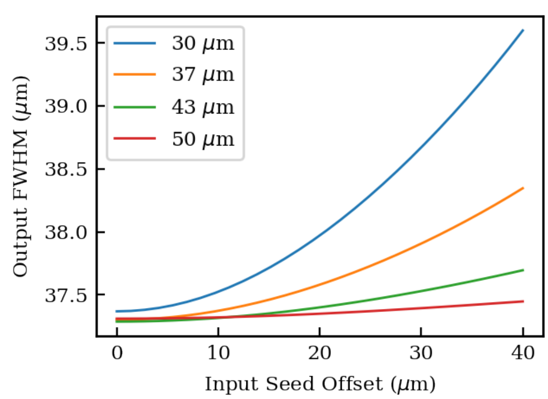

Figure 1: The dependence of the output fwhm of the radiation on the input transverse offset is shown, with labels indicating the input fwhm.

The second approximation, justified retroactively in the next section, is that the matrix values may be approximately taken as not varying with the input mode parameter Q. This is fundamentally enabled by the fact that these matrix values vary relatively slowly with the input beam parameters within certain qualitatively defined regions of the parameter space, as we will discuss in more detail in the next section. Quantitatively, we may take advantage of these approximations by first noting that

$$k_r A_{\text{emp}} + Q_i B_{\text{emp}} = \frac{k_r}{Q_f}\frac{Q_i x_{0,i}}{x_{0,f}}$$
$$k_r C_{\text{emp}} + Q_i D_{\text{emp}} = \frac{Q_i x_{0,i}}{x_{0,f}} \tag{5}$$

By performing a linear fit to the right-hand side of these expressions, as obtained by a large sample of simulations with different input values of Q, one may extract an empirically-derived, Q- and x_0-independent ABCD matrix for the FEL which models the transverse dynamics with an accuracy comparable to that achieved by the refractive index model presented in [7].

EXAMPLE CALCULATION WITH NUMERICAL BENCHMARKS

To both demonstrate and benchmark the empirical ABCD approach we have used the LCLS-II-HE parameters most relevant to the XRAFEL project at SLAC [7, 9]. These are summarized in Table 1. For different simulations, we vary the input transverse mode parameters: in particular the Rayleigh length and free space waist location. The radiation seed is initialized in every case with a 20 μm transverse offset. Since the refractive index model operates only before saturation, the magnitude of the input seed power does not impact the simulation results.

Table 1: Parameters for Numerical Benchmarks

Parameter	Variable	Unit	Value
E-Beam Energy	γmc^2	GeV	8.003
E-Beam Emittance	$\epsilon_{nx,ny}$	nm rad	350
Energy Spread	σ_γ/γ	10^{-4}	0.875
Pierce Parameter	ρ	10^{-4}	5.45
E-Beam rms Size	$\sigma_{x,y}$	μm	17
Undulator Period	λ_u	cm	2.6
Seed Wavelength	λ_{rad}	Å	1.261
Detuning	$\Delta\nu$	10^{-4}	-8.13
Undulator Length	L_u	m	23.66

Following the procedure described in the previous section, we ran a total of 600 simulations and obtained an ABCD matrix empirically derived from linear fits to these simulation outputs. In total the input Rayleigh range was scanned from 10 m to 150 m, and the free space waist from 0 m to 70 m.

We demonstrate the accuracy of the method in Figs. 2 and 3. In the former, we show the predicted output radiation centroid as produced from the refractive index code (x-axis) and from the empirical ABCD matrix with Eq. (4). The top plot shows a direct comparison of the two alongside the optimal line $y = x$ which would correspond to a perfect one-to-one correspondence between the two approaches. There is some deviation, which we show in a more quantitative way in the bottom figure. Across this range the root-mean-square error between the two results is roughly 1%.

Figure 3 displays analogous data for the full-width at half-maximum of the field profile. In this case, the root-mean-square error is just 0.08%. We note in particular that, on account of the strength of FEL guiding effects, the absolute spread in final predicted fwhm values is not very large. Even with these deviations being a relatively weak result of

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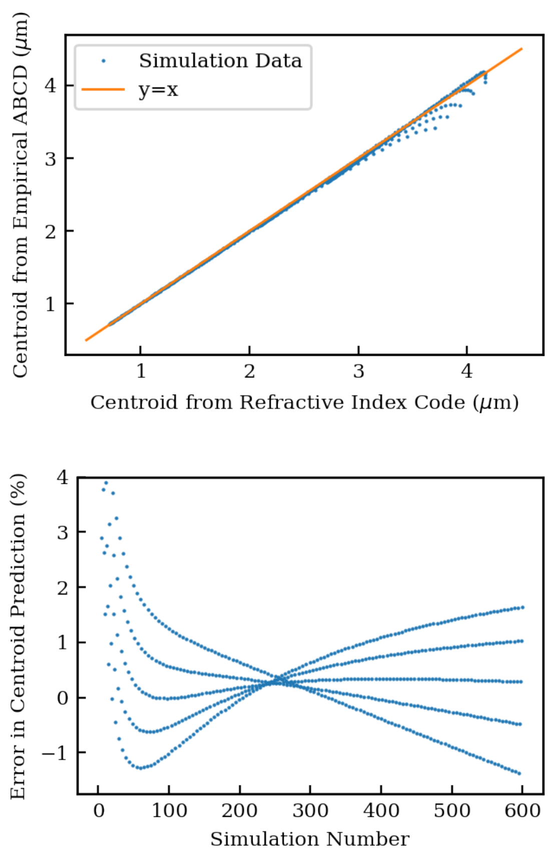

Figure 2: The output radiation centroid from the empirical ABCD matrix is compared to the output of the refractive index approach. The top plot shows a high degree of agreement between the two approaches, and the bottom plot reveals a root-mean-square error of just 1%.

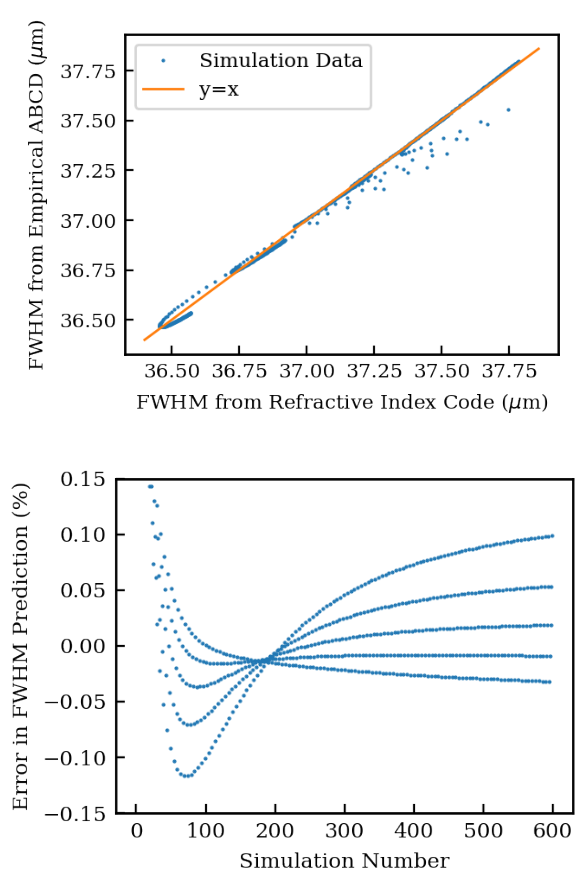

Figure 3: The output radiation full-width at half-maximum from the empirical ABCD matrix is compared to the output of the refractive index approach. The top plot shows a high degree of agreement between the two approaches, and the bottom plot reveals a root-mean-square error of just 0.08%.

differing initial conditions, the ABCD matrix approach can capture these slight nuances with high accuracy.

The largest errors occur for smaller input Rayleigh lengths, which is to be expected and allows us to clarify the range of validity of this approach. As is well-known from FEL theory, and has been reproduced in Fig. 3, the FEL always tends to guide the radiation mode to a particular mode size referred to as the fundamental high gain mode. In this case we can see that this corresponds to a fwhm of roughly 37 μm, or equivalently a Rayleigh length of 12 m. As a result of this trend towards a roughly initial condition-independent mode size, the fundamental properties of the guiding process change as the input beam is brought closer to the expected high gain fundamental mode. The initial focusing effects are naturally reduced, as the system is already close to the equilibrium it seeks out, so the ABCD elements change correspondingly. Subsequently, since most of the data points we performed our fit to were for input mode sizes which were large relative to the guided mode, the data points which have a size comparable to the guided mode are less accurately represented by the fit. For these reasons, the empirically fit ABCD values should not be extrapolated outside of the qualitative portion of the parameter space which they were fit to.

SUMMARY

We have presented a simple method for analyzing transverse dynamics in free-electron lasers using an empirically-derived ABCD matrix. The model is naturally extremely fast, and has a broad range of input mode parameters over which it may be considered accurate. With it, both the radiation centroid and the full-width at half-maximum of the transverse field profile can be tracked, which makes this naturally useful for XRAFEL studies including optimization and error tolerance.

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